ELEMENTARY

MECHANICS

BY

KAVASJEE D. NAEGAMVALA.

FIFTH EDITION.

BOMBAY:

RADHABAI ATMARAM SAGOON.

1902.

(Price Rupee One.)

ELEMENTARY

MECHANICS.

(*ILLUSTRATED.*)

BEING

A FIRST MANUAL OF STATICS

as required for the Matriculation Examination of the University of Bombay.

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PART I.—PRELIMINARY NOTIONS.

CHAPTER I.

Force, forces acting in a straight line.

1. Looking round about us we perceive that all bodies are either moving or at rest. To set a body in motion and to keep it moving, some external agency must necessarily be employed. On the other hand with regard to a body at rest, it is not so self-evident that the body is under a similar external influence, but that such is really the case will become clear if we consider some typical examples. Take, for instance, the case of a book let go from the hand and falling on a table. What brings it to rest and why does it afterwards remain at rest? The answer is clear;—it was falling under the attraction of the earth, but when it approached the table, the resistance of the table stopped its motion. A stone hanging from a rope is another instance in point. If we cut the rope, the stone at once begins to fall. What then prevented it from falling before? There was something evidently in the rope, its tension, which kept the stone stationary.

2. In nature, therefore, there is always some agency which keeps a body in motion or at rest, or else starts a body into motion or brings a moving body to rest; this is called force.

Def.—Any **cause** which **changes or tends to change the state** of rest or of motion of a body is called **force.**

3. A force, therefore, implies one of the following four things, *viz* :—

(1) that by its agency a body is *set in motion* ; or

(2) that by it an ineffectual *attempt* is made *to put* a body *in motion* ; or

(3) that by its means a moving body is *brought to a standstill* ; or finally

(4) that an ineffectual *attempt* is made *to stay the progress* of a moving body.

For instance, the muscular exertion employed *to move* a body originally at rest, or any other cause producing a like effect, such as the agency of a steam-engine, is force. Even when the body is not moved by such agency, as for example, when we press against a masonry wall but cannot move it, we say that we employ force, because there is an *attempt made to move* the body. When we catch a cricket ball, we wholly *change its state of motion into that of rest*, and this is done by employing the muscular force of the hand ; but when we pull a moving carriage from behind we can very slightly lessen its speed, and thus can only partially change its state of motion, but nevertheless, as we make an *attempt to stop* it, we are said to have employed force.

4. The science which investigates the action of forces is called **Dynamics.*** It has more often, though erroneously, been styled *Mechanics.* It is devided into two branches, *viz.*, **Kinetics** and **Statics.**

In **Kinetics** the action of forces in producing or in changing motion is considered ; while in **Statics** the action of forces in maintaining rest or preventing change

* From Gr. *Dynamis*, power of force.

of motion, in other words, the equilibrium or the balancing of forces is investigated.

Def.—**Statics** is that branch of the more general **science** of Dynamics which treats of the **conditions** under which several **forces** acting on a mass of matter maintain it in the state of **equilibrium.**

5. The forces, that we are considering, act on the various kinds of *matter* of which all substances in the universe are composed. Matter has an innate power of resisting external influences, so that everybody, *as far as it can*, maintains its previous state either of rest or of motion; this is called the *inertia* of matter. A definite portion of matter is called a *body* and an exceedingly small portion of it is called a *particle*. The *mass* of a body is the amount of matter contained in it. The **unit of mass** generally employed is the pound, and is defined to be the mass of a certain **standard piece of platinum** recognised as such by Act of Parliament. A body, therefore, is said to have a mass of ten pounds or twenty, when it contains ten or twenty times as much matter as the standard platinum piece above referred to.

Def—Anything to which a force can be applied and which offers resistence to it is **matter.**

Def.—The passive resistence offered by matter to force is called its **inertia.**

Def.—A portion of matter limited in all directions is called a **body.**

Def.—A body whose size is so small that it may be regarded as a quantity of matter collected at a single point is called a **particle.**

Def—The quantity of matter in a given body is its **mass.**

6. **Gravity.**—The most general force pervading the universe is the force of gravitation or gravity. Owing to it every object has a tendency to fall to the surface of the earth, and the force with which a body is so attracted is called its *weight*. To oppose this tendency a certain amount of muscular exertion or some other force is necessary, consequently the muscular exertion or any other agency by which we prevent a certain mass from reaching the surface of the earth is also a right measure of its weight.

As gravity is the force best known to us, we shall choose as our unit of force the force of gravity acting on one pound of matter, *i. e.*, the weight of one pound.

Def.—The force with which a body is attracted by the earth to itself is called its **weight.**

Forces in statics do not produce motion, and they are measured by the masses they would support against the force of gravity or what is the same thing, by the weights they would sustain. For instance, by the expression 'a force of 5 lbs.' we mean 'a force equal to the weight of 5 lbs ,' or still more correctly 'a force equal to the force of gravity acting on a mass of 5 lbs.'

Def.—**Unit Force** is the force which just supports a mass of one pound against the force of gravity, the unit of mass being one pound.

7. **Particulars necessary for defining a force.**

Three particulars must be attended to, in order to

completely define a force or to estimate the effect of a force upon a body ; these are called its **elements.** They are :—

(i) the **point of application** of the force,

(ii) the **direction** in which the force acts, and

(iii) the **magnitude** of the force.

All these **data** can be specified **by a straight line** of finite length, because

(i) a straight **line** can be **drawn from any point,** and, therefore, if one be drawn from the point of application of a force, it will **represent** that force with respect to its **point of application** ;

(ii) again a straight **line** can be **drawn in any direction**, and, therefore, if one be drawn from the point of application in the direction in which the force tends to produce motion, that straight line will **further represent** the force with respect to **its direction** ; and lastly

(iii) this straight **line** can be **drawn of such a length** as to contain as many units of length as the force contains units of force ; therefore, when it is so drawn, it will **also represent** the **magnitude** of the force.*

An arrow-head is used to denote the **sense** of the direction in which the force acts. Thus, in the first

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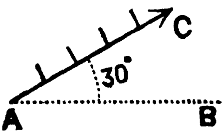
* *Scale of representation of forces.*—For the length of a straight line to correctly represent the magnitude of a force, we should at first fix upon a certain length to represent a unit of force. Thus if a line one inch long represents a force of 1 lb., then a line two inches long will properly represent a force of 2 lbs. But it is not necessary to select a *unit* of length to represent a unit of force; we may fix upon half an inch to represent a force of 1 lb., then an inch length will represent a force of 2 lbs., and so on.

following figure the force acts from A to C and in the second figure the two forces act towards A. The sense of the direction is also indicated by the order of the letters used in naming the line which represents the force. Thus AC represents a force acting from A towards C (first figure), while CA represents a force acting from C towards A (second figure).

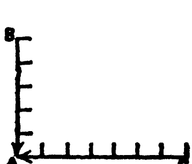
The pictorial method of representing the elements of a force by a straight line is called the **graphical method** of representing forces.

ILLUSTRATIONS :—

(1) Suppose it is required to represent a force of 5 lbs. acting on a point but away from it at an inclination of 30° to the horizon. Then let A be the point and AB the horizontal straight line; draw a straight line AC at the angle of 30° to the horizontal line AB, containing five units of length with arrow-head as shown in the figure, then the line AC will completely represent the given force; because (i) its **point of application** is A, (ii) its **direction** is given by the angle BAC and the arrow-head and (iii) its **magnitude** is represented by the length of the line AC.



(2) Again, suppose it is required to represent two forces P and Q of **5 lbs.** and **7 lbs.** respectively, applied to a point at **right angles** to each other, both **acting towards the point.** Then taking any line we please to represent the unit of length, we draw two lines AB and AC at right angles to each other, the one containing our unit of length



five times and the other containing it seven times with arrow-heads as shown in the figure. Then the lines BA, CA will properly represent the two forces acting at the point A, because (i) their **point of application** is given by A, (ii) their mutual **directions** by the angle BAC and the arrow-heads, and (iii) their respective **magnitudes** by the lengths of the lines BA and CA respectively.

EXAMPLES.

1. If a force of 5 lbs. be represented by a line 10 inches long, what force will be represented by a straight line 1 foot 6 inches long?

   10 in. represent 5 lbs.,
   ∴ 1 in. represents $\frac{1}{2}$ lb.,
   and 1 ft. 6 in. = 18 in.
   represent 18 times $\frac{1}{2}$ or 9 lbs.

2. Three lines 2 in., $1\frac{1}{2}$ in. and 6 in. drawn from a point A in given directions represent three forces; the smallest force is 3 lbs., what are the magnitudes of the others? *Ans.* 4 lbs. and 12 lbs.

3. If a force which can just sustain a weight of 5 lbs. be represented by a straight line whose length is 1 foot 3 inches, what force will be represented by a straight line 2 feet long? *Ans.* 8 lbs.

4. If a force of P lbs. be represented by a straight line a inches long, what force will be represented by a straight line b inches long? *Ans.* $\frac{b}{a}$ P lbs.

5. Three forces are proportional to the sides of a triangle whose sides are 2 in., 3 in. and 4 in.; if the greatest force is 6 lbs., what are the others? *Ans.* 3 lbs. and $4\frac{1}{2}$ lbs.

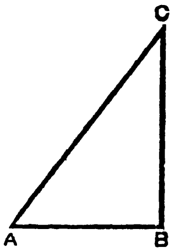
6. How would a force of a kilogramme be represented if a straight line a centimetre long were the representation of a force of one gramme? *Ans.* 1000 cms.

7. How would a force of 9 cwts. 3 qrs. be represented if a straight line ·05 metre long represents a force of 1 lb.? *Ans.* 54·6 metres.

8. If a foot be the unit of length and a pound the unit of force, find the lengths of the lines which will represent forces of 10 grains

10 ounces and 1 ton respectively. *Ans.* 0·01714 in., 7·5 in., 2240 ft.

The sides of a right-angled triangle represent three forces; the two sides containing the right-angle are in the proportion of 3 to 4. If the smaller side represents a force of 12 lbs., what must be the other forces equal to?



Let ABC be the triangle of which angle ABC is the right angle,

and let AB : BC : : 3 : 4;

then as $AC^2 = AB^2 + BC^2$ (Eu. I., 47.)

$= 3^2 + 14^2 = 25,$

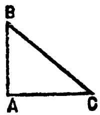
and $AC = 5$;

and the three forces represented by AB, BC and AC are respectively as 3, 4 and 5.

The force represented by AB, the smallest of the three sides, is however given to be equal to 12 lbs.; therefore, the force represented by BC is equal to 16 lbs., and by AC equal to 20 lbs.

10. The sides of a right-angled triangle represent three forces and the two sides containing the right-angle are 5 and 12 inches long. If the longer side represents a force of 84 lbs., what must be the other forces equal to? *Ans.* 35 lbs. and 91 lbs.

11. In a right-angled triangle the sides are 6 and 8 inches and a force of 50 lbs. is represented by the hypotenuse; what forces are represented by the sides? *Ans.* 30 lbs. and 40 lbs.

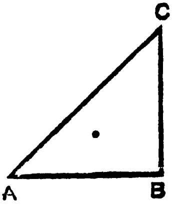


12. A force of 6 lbs. acting towards the North and of 8 lbs. acting towards the East are represented by straight lines A B and A C respectively: determine the magnitude of the force represented by the straight line BC.

In the triangle ABC, the angle BAC is a right-angle, being the angle between N and E and A B = 6 lbs. and A C = 8 lbs.,

$\therefore$ $BC = \sqrt{6^2 + 8^2} = 10$ lbs.

13. The sides of a right angle isosceles triangle represent three forces, and the greatest force is 20 lbs: find the other forces.



Let ABC be the triangle and AC its hypotenuse.

Then $AC^2 = AB^2 + BC^2$ (Eu. I., 47.)

and $AB = BC$

$\therefore AC^2 = 2\,AB^2$

and $AC = AB\sqrt{2}$

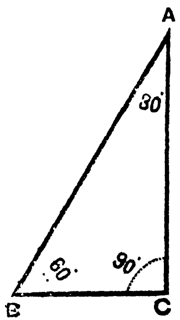
and $\qquad$ AC

$= \frac{20}{\sqrt{2}} = 10\sqrt{2}$ lbs.

14. The sides of a right angled isosceles triangle represent three forces. The least force is $5\sqrt{2}$ lbs., find the greatest force. *Ans.* 10 lbs.

15. Two forces each of $15\sqrt{2}$ lbs., one towards the north and the other towards the west are represented by AB and AC respectively. What force is represented by BC ? *Ans.* 30 lbs.

16. Two angles of a triangle are of 60° and 90°, and the three sides of the triangle represent three forces; if the force represented by the side opposite the third angle is 15 lbs., determine the other two forces.



The third angle of the triangle is of 30°.

Let ABC be the triangle with angles $ABC = 60^\circ$, $BCA = 90^\circ$ and $BAC = 30^\circ$.

Then AB : BC : CA

$:: \; 1 : \frac{1}{2} : \sqrt{\frac{3}{4}}$ (*vide* Appendix B),

$\therefore \; AB = 2\,BC$

and $CA = \sqrt{3}\,BC$;

but $BC = 15$ lbs.,

$\therefore \; AB = 2.\,15 = 30$ lbs.,

and $CA = 15\sqrt{3}$ lbs.

17. If the sides of a square represent forces of 5 lbs. each, determine what force is represented by the diagonal. *Ans.* $5\sqrt{2}$ lbs.

18. If in a square the diagonal represents a force of 10 lbs., find what forces will be represented by the sides. *Ans.* $\frac{10}{\sqrt{2}} = 5\sqrt{2}$ lbs.

19. ABC is an equalateral triangle and AD is drawn perpendicular to BC. If AD represents a force of 3 lbs., what force is represented by AB ?

Let B D be equal to unity.

As $BD = DC$ (Eu. I., 26.)
$\therefore$ BC is twice $BD = 2$,
and also AB is twice $BD = 2$.
Now $AB^2 = AD^2 + BD^2$ (Eu. I., 47.)
$\therefore$ $AD^2 = AB^2 - BD^2$
$= 2^2 - 1^2 = 3$
and $AD = \sqrt{3}$.
Hence $AD : AB :: \sqrt{3} : 2$
$:: 3 : 2\sqrt{3}$. *Ans.* $2\sqrt{3}$ lbs.

20. The perpendicular drawn from the vertex of an equilateral triangle to the base represents a force of $5\sqrt{3}$ lbs., what force is represented by each side of the triangle? *Ans.* 10 lbs.

21. The side of an equilateral triangle represents a force of 30 lbs.; what force is represented by its perpendicular? *Ans.* $15\sqrt{3}$ lbs.

22. Equal forces each of 8 lbs., inclined at an angle of 60° to each other are represented by lines AB and AC; what is the magnitude of the force represented by the line joining B and C? *Ans.* 8 lbs.

23. The two diagonels of a rhombus represent two forces of 12 lbs. and 16 lbs. What force will be represented by each side?

(*Note.*—The diagonals of a rhombus bisect each other at right angles.) *Ans.* 10 lbs.

8. Various kinds of forces. Gravity further considered.—We have already mentioned the all-pervading force of gravity. It seems to 'act at a distance,' *i. e.*, without the bodies being necessarily in contact and without the intervention of any visible agengy; such is the case when we see a stone falling towards the earth. The direction in which gravity acts at any place may be found by letting a body fall freely or by noting the position of a plumb-line, which consists of a heavy weight suspended by a string. This path or direction is always straight; it is pointed towards the centre of the earth and is called the vertical line or simply the *vertical* at that place. All verticals, therefore, converge towards the earth's centre, but owing to

the large size of the earth, its average radius being nearly 4,000 miles, verticals at places within a few miles may be practically taken as parallel to one another; hence it is that the directions of two weights hanging in a room are to all intents and purposes parallel.

Def.—The direction of the plumb-line or the direction in which a particle would fall freely at any place is called the **vertical** at that place.

Def.—A plane perpendicular to the vertical line is said to be **horizontal.**

9. Static forces. Pressure and tension.—Static forces, as already remarked, do not cause motion; they are prevented from doing so by some kind of resistance, and the effect they produce on a body is either pressure or tension.

For instance, take a book lying on a table. We know that the book is acted upon by the omni-present force of terrestrial attraction or gravity tending to move the book vertically downwards, yet the book is at rest. There must, therefore, be an **equal and opposite force** exerted by the table on the book against that of gravity; this is the **reaction** of the table, while the force with which the book presses upon the table owing to gravity is the **measure of the static force** exerted by it; it is called its **pressure** and its intensity is measured by the weight of the book. Similarly, if the book be placed on a lump of india-rubber, the india-rubber yields a little under the *weight* of the book, and then the book and the piece of india-rubber remain relatively at rest; the mutual forces exerted by them then attain the state of equilibrium,—the pressure of the book being equal to the reaction of the india-rubber.

Def.—**A pair** of forces which always **go together** is called a **stress.**

Def.—The **effect** which a stress produces in a body before it is ruptured is called a **strain.**

Def.—If two bodies be forced against one another and no motion is produced, the stress is called **pressure** or *action and reaction*; these **forces are equal and opposite, and act towards each other.**

If, however, the book, previously considered, be suspended by a string and thus prevented from falling under the force of gravity, it will **stretch** the string and keep it tight, or if it be suspended from a piece of elastic or from the hook of a spring-balance, the elastic or the spring will be strained or deformed a little, and then the book and the elastic or the spring, as the case may be, will remain relatively at rest. In these three cases the force of gravity will act downwards, while the equal and opposite force exerted by the string or the elastic or the spring of the balance will act upwards. The forces, as in pressure, will be equal and opposite, but they will be acting away from each other; this is called **tension**. In the same manner when a rope is pulled in a tug-of-war, tension is produced on the rope by each of the parties at the two ends.

Def.—When an attempt is made to pull a body the stress is called **tension** and is measured by the strain produced on the medium (string, chain, spring, &c.,) through which the body is pulled; **the forces are equal and opposite, but act away from each other.**

We thus notice that the **different ways** in which forces are brought into play are **three**, *viz.*:—

(1) *attraction*, in which the particles are not originally in contact nor is there any tangible medium intervening between them, but the tendency of

the forces called into play is to bring the particles together,—such is the force of gravity;

(2) *pressure*, or action and reaction, in which the particles between which the forces are exerted are close together and have a tendency to come still closer, and

(3) *tension*, in which the particles between which the forces are excrted are very near and have a tendency to be pulled away from each other.

10. Equilibrium of forces acting in the same straight line.

If we have two forces (*a*) equal in magnitude, (*b*) acting at the same point, and (*c*) in opposite direcions, then it is clear that the motion which one force tends to impart is exactly the reverse of that which the other tends to produce and the body cannot move, as there is no reason why it should do so in one direction rather than in the other ; the body remains at rest, and the two forces are said to be in *equilibrium* or to *balance* one another.

Algebrically—

$$P = Q$$
$$\text{or } P - Q = 0.$$

Signs as applied to forces.—It is convenient, to assign *opposite signs to opposite forces*; thus, if forces in one direction be designated by the sign +, those in the opposite directions must have evidently the sign — attached to them and it will be seen that then the algebraical sum* of equal opposite forces will be zero.

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* The sum of any number of quantities with their proper signs attached to them is called the *algebraical sum*.

If instead of two, there are several forces acting at a point and all in the same straight line, and if they do not produce any motion, it is evident that the tendency to motion in one direction must be counterbalanced by that in the opposite direction; *i.e.*, the sum of the forces in one direction must be equal to the sum of the forces in the opposite direction : in other words, the **algebraical sum** of the forces must **vanish.**

If P_1, P_2, P_3,......P_m be the several forces in one direction and Q_1, Q_2, Q_3,.........Q_n be the several forces in the opposite direction, then the condition of equilibrium is that $P_1 + P_2 + P_3 + \ldots\ldots + P_m = Q_1 + Q_2 + Q_3 + \ldots + Q_n$; or that their *algebraical* sum must be zero *i.e.*, $(P_1 + P_2 + P_3 + \ldots\ldots + P_m) - (Q_1 + Q_2 + Q_3 + \ldots\ldots + Q_n) = 0$.

Def.—When two or more forces act upon a body and are so arranged that the body remains at rest, the forces are said to be in **equilibrium.**

Def.—A number of forces considered together, but apart from *all other* forces, is called a **system of forces.**

Def.—**Equal forces** are those which have the same magnitude or which produce the same effect on a given body.

Def.—The direction in which a force acts is called its **line of action.**

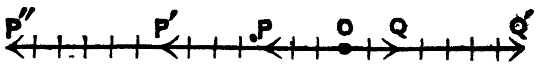
11. Resultant and component of forces.

If *two equal forces* P and Q act on a particle O in opposite directions *in the same straight line*

their combined effect will be nil. If P be + 4 then Q will be equal to − 4 and their sum zero.

If, however, *one of them be greater than the other* (*e.g.*, $P=6$ and $Q=4$) then they will produce the same effect as one single force of two units $(6 - 4 = 2)$, and in the direction of the greater force P.

And lastly, if there be two *systems of forces acting opposite to one another*, the joint effect will be the *excess* of the one system over the other in the direction in which the larger system acts ; in other words, it is equal to the *algebraical* sum of all the forces taken together, the forces on one side being taken as positive and on the other as negative. For example, in the figure giving proper signs $P = 3$, $P' = 4$ and $P'' = 6$, while $Q = -2$ and $Q' = -5$, and therefore the resultant is equal to $3 + 4 + 6 - 2 - 5 = 6$ in the direction of P, P′, and P″.



If all the *forces*, however, *act in the same direction*, then the total effect will be equal to that of the forces added together. In the figure the total effect is equal to twelve units. The single force whose combined effect is equal to all the separate forces is called the *resultant.*

We have seen above that when several forces act on a particle, they can all be removed and replaced by a single force, the resultant. The converse operation can also be performed and we can replace a single force by several which produce together the same effect as the original single force. For example, if a man is dragging a heavy load with a rope, exerting through the medium of the rope a force of 50 lbs., we can remove him from the work and set instead, say,

five boys to pull at the rope ; the first of them may exert a force of 15 lbs., the second of 12 lbs., the third of 10 lbs., the fourth of 8 lbs., and the fifth and last of 5 lbs., and the five boys together will produce the same effect as the man alone did previously. When a single force is thus replaced by several forces all together producing the same effect, the several forces are called the **components** of the original force and the process of thus replacing one force by several is known as the **resolution** of forces.

Def.—When **any number of forces** act simultaneously on a particle, these forces may be **reduced to one,** that is, they may be replaced by a single force **producing** precisely the **same effect** as them all ; this single force is called the **resultant.**

Def.— The **process** by which a **single force** is found **equivalent to a number** of forces acting simultaneously is called the **composition** of forces.

Def.—The two or more **forces** whose simultaneous **effect** is the same as that **of a single force** are called the **components** of that single force.

Def.—The **process** by which a **single force** is **replaced by a number** of forces acting simultaneously and producing the same effect is called the **resolution** of forces.

It must be distinctly borne in mind that *resolution* does not mean finding the resultant, but it means the *resolving* or breaking up of a single force ; similarly, *composition* does not mean finding the components, but it means the *compounding* or putting together of several forces to get one.

EXAMPLES.

1. **Find the resultant of forces 3 lbs., 4 lbs. and 5 lbs. acting on a particle towards the North and of 2 lbs. and 8 lbs. acting on the same particle towards the South.**

We have $3+4+5=12$ lbs. in one direction (the north) and $2+8=10$ lbs. in exactly the opposite direction; hence the resultant is $12-10=2$ lbs. towards the North.

2. Find the resultant of forces 2 lbs., 8 lbs. and 6 lbs. acting upwards and 5 lbs., 8 lbs. and 4 lbs. acting downwards. *Ans.* 1 lb. downwards.

3. Find the resultant of forces + 2 lbs, + 6 lbs, + 8 lbs., - 3 lbs., - 5 lbs., - 15 lbs. all acting towards the East. *Ans.* 7 lbs. towards the West.

4. In a tug-of-war 14 boys pull on one side and 20 girls on the other. If each boy pulled with a force of 30 lbs. and each girl with a force of 25 lbs., which side will win? *Ans.* The girls, by 80 lbs.

5. ABCD is a straight line; find the resultant of the forces represented by AB, BC, CD and DB. *Ans.* A force equal to AB.

6. Two sailors pull one at each end of a rope with a force of 100 lbs. each; required the tension in the rope. *Ans.* 100 lbs.

7. A bullock cart is ascending a steep hill and it requires a pull in front of 300 lbs. to keep it from sliding down the slope. Two men push it from behind each exerting a force of 50 lbs., while one out of the two bullocks is pulling it with a force of 125 lbs., what pull is the other bullock exerting when the cart is just stationary?

The push from behind has the same effect as the pull in front; hence the force exerted by the two men and the first bullock is equal to $50+50+125=225$ lbs. It requiries, however, a force of 300 lbs to keep the cart stationary; therefore, the second bullock must be pulling with a force sufficient to make up 300 lbs., *i. e.*, $300-225=75$ lbs.

8. Arrange forces of 2 lbs., 4 lbs., 5 lbs. and 8 lbs. acting in the same line so that their resultant may be as small as possible. *Ans.* The forces of 4 lbs. and 5 lbs. should be opposed to those of 2 lbs. and 8 lbs.

9. Can forces of 3 lbs., 8 lbs., 7 lbs. and 2 lbs. acting on a particle be so arranged as to be in equilibrium?

To have the forces in equilibrium their algebraical sum must be zero and by trial it will be seen that the sum of $+3, +7, -8$, and -2 is zero; therefore when the forces of 3 lbs. and 7 lbs. act opposite to those of 2 lbs. and 8 lbs. the system will be in equilibrium.

10. Find the resultants of forces 3 lbs., 4 lbs., and 5 lbs., according to the different possible arrangements of them; all three acting in the same straight line. *Ans.* 12 lbs., 6 lbs., 4 lbs., 2 lbs.

11. Arrange nine forces of 1, 2, 3, 4, 6, 8, 9, 12 and 15 lbs. acting on a particle in directions north, east, south and west in such a manner that the particle may not move.

Ans. The four sets are:—

$$1 + 2 + 12 = 15 \text{ lbs.}$$
$$4 + 3 + 8 = 15 \text{ lbs.}$$
$$6 + 9 = 15 \text{ lbs.}$$
and 15 lbs.

12. If three forces represented by the numbers 1, 2, 3 acting in one place keep a particle at rest, show that they must all act in the same straight line.

13. Three forces of 10 lbs., 20 lbs., and 40 lbs. act on a body, their directions being all in the same straight line; find a fourth force which will balance them. *Ans.* 70 lbs. in the opposite direction.

14. Three forces of 4 lbs., 9 lbs., and 5 lbs. acting on a point in a straight line are in equilibrium, how are they directed?
Ans. Forces of 4 lbs. and 5 lbs. are opposite to that of 9 lbs.

15. A, B, C and D are points on a straight line; show that for any arrangement whatever of the points, forces represented by A B, B D, D C and C A are in equilbrium.

16. If three forces P, Q, and R acting on a particle in a straight line keep it in equilibrium, what is the resultant of the two forces P and R? *Ans.* Q equal to P+R and acting in the opposite direction.

17. Two forces act on a particle and their greatest and least resultants are 72 lbs. and 56 lbs.; find the forces.

$$P + Q = 72,$$
$$P - Q = 56,$$
$$2P = 128,$$
and $P = 64$ lbs.,
and $\therefore Q = 8$ lbs. *Ans.* 64 lbs., and 8 lbs.

18. Twenty-two times the least resultant of two forces is 110 lbs. and three times the greatest is 105 lbs., find the two forces.
Ans. 20 lbs., and 15 lbs.

19. The greatest resultant of two forces is 450 lbs., and one force exceeds the other by 200 lbs. Find the two forces. *Ans.* 325 lbs. and 125 lbs.

20. The greatest resultant of two forces is 1000 lbs., and one force is three times the other. Find the two forces. *Ans.* 250 lbs. and 750 lbs.

12. Superposition of forces.—If in the several instances of forces acting opposite to one another mentioned in the preceding article we add equal forces to each side or remove them from either side, the final result will in no way be affected ; this is evident, because the new forces added or removed form by themselves a system in equilibrium. There is no *practical* utility in thus adding or romoving equal forces, but the conception is of some use in solving problems. This principle is called the **principle of the superposition of forces.**

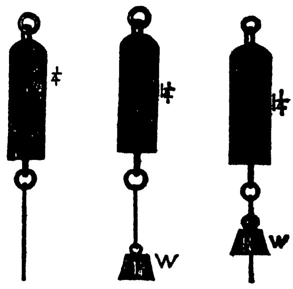
Def. If a system of forces in equilibrium be added to or removed from another system already in equilibrium, the new system will still be in equilibrium. This principle is known as **superposition of forces.**

13. Transmissibility of forces.

Def.—If a force acts on a body, its **effect** will **not** be **altered if the force be transferred to any other point** provided that (1) this point is rigidly connected with the body, and (2) is in the original line action of the force. This principle is called the **transmissibility of forces.**

The truth of this is borne out by our every-day experience. If we push a book away from us with the end of a stick, we apply a force to the book and it does not matter whether the stick be long or short; the distance between the hand and the book may vary but as long as the 'line of action' is not changed and the hand is in 'rigid' contact with the book through the medium of the stick, the effect remains the same. Again, if two persons or sets of persons pull against one another, as in a tug-of-war, holding to the extremities of a rope, the result is the same no matter what the length of the rope be, if we neglect the weight of the rope.

This may be further illustrated by an experiment. From the hook of a spring-balance hang a cord weighing (say) four ounces, then the pointer will indicate four ounces; next suspend a pound weight from the other end of the cord, then the pointer will come down to $1\frac{1}{4}$ lb., raise the weight up the cord and let it hang from different points on the cord, but this will not alter the weight indicated by the balance. This follows from the principle of transmissibility, because the weight, (1) being rigidly connected with the balance and (2) being also in the original line of action, *viz.*, the vertical, produces the same effect on the spring of the balance whatever be the point on the cord from which it is suspended.



EXAMPLES.

1. A string sustains a weight of 15 lbs. at its extremity and a weight of 10 lbs. at its middle point; find the tension in the two halves of the string.

Let ABC be the string and B its middle point. The point B supports the two weights hanging below, *viz*., 10 + 15 = 25 lbs., this is the tension in AB; while the tension in BC is equal to the weight supported by it, *i. e.* 15 lbs.

Ans. Tension in AB = 25 lbs.

Tension in BC = 15 lbs.

2. If in the above example the tension in the lower half of the string is graphically represented by a line 3 inches long what length will represent the tension in the upper half? *Ans.* 5 inches.

3. A string AD is suspended at A; below it at point B a weight of 3 ozs. is attached, at C another of 4 ozs. and at D again another of 5 ozs. Find the tension in each part of the string. *Ans.* AD = 12 ozs. BC = 9 ozs., CD = 5 ozs.

4. One end of a uniform chain 2 feet long and weighing 9 lbs. is fastened to a hook and hangs vertically. What is the tension at a distance of 8 inches from the upper end? *Ans.* 6 lbs.

5. A uniform chain 2 feet long and 3 lbs. in weight has weights 20 lbs. and 21 lbs. attached to its ends and is at rest when hung over a smooth peg; find the lengths with which it is divided by the peg.

Let x = length in feet of one portion, then $2 - x$ = length in feet of the other portion.

As the whole chain weighs 3 lbs; one foot of it weight $\frac{3}{2}$ lbs., and x feet weighs $\frac{3}{2}x$ lb. Similarly $(2 - x)$ feet weighs $\frac{3}{2}(2 - x)$ lbs.

Therefore the total weight on one side of the peg is $(\frac{3}{2}x + 20)$ lbs. and on the other $\frac{3}{2}(2 - x) + 21$ lbs.

As these weights are in equilibrium,

$$\therefore \quad \frac{3}{2}x + 20 = \frac{3}{2}(2 - x) + 21$$

$$6x = 8$$

$$x = \frac{4}{3} \text{ feet.}$$

and $\quad 2 - x = \frac{2}{3}$ feet. *Ans.* $1\frac{1}{3}$ ft. and $\frac{2}{3}$ ft.

6. A uniform chain 15 ft. long and 5 lbs. in weight has weights of 5 lbs. and 8 lbs. attached to its ends and rests in equilibrium over a smooth peg. Find the lengths of the two parts of the chain on each side of the peg. *Ans.* 12 ft. and 3 ft.

7. A chain 12 ft. long and 3 lbs. in weight passes over a smooth peg, which divides it into two parts in the ratio of 1 : 5. If a weight of 8 lbs. is attached to the shorter part of the chain, find what weight must be attached to the longer part, so that the chain may be at rest. *Ans.* 6 lbs.

QUESTIONS.

1. What is *force*? Mention the different effects that force can produce. Define *unit of force.*

2. What is the science of *Dynamics*? Distinguish between *Statics* and *Kinetics.*

3. What do we call *matter*? Define *inertia*; also a *particle, a body* and *mass*. What is a *unit mass*?

4. Name the *elements of a force.* Why are they necessary in fully determining a given force?

5. How may forces be represented graphically by straight lines?

6. Why is a 'scale' necessary in representing forces graphically?

7. What kind of force is *gravity*? What are its effects?

8. Define *vertical* and *horizontal* directions. Why are neighbouring 'verticals' parallel to one another?

9. When is a force said to be in the *static condition*?

10. Define *stress* and *strain.*

11. Explain fully the dual nature of force with examples.

12. Distinguish between *pressure* and *tension* and give examples.

13. What is the *line of action* of a force? What is a *system of forces*? When are forces said to be *equal*?

14. Name the condition of equilibrium of a system of forces acting in the same line.

15. What is the utility of *signs* in designating the direction of forces?

16. Define *resultant* and *components.* Distinguish between *resolution* and *composition* of forces.

17. What is meant by the *super-position of forces.*

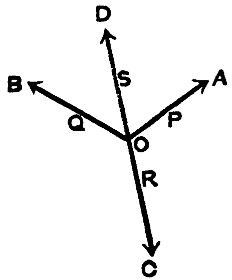
18. What is the principle of *transmissibility of forces*? How many it be verified?

CHAPTER 11.

Forces acting at an angle. Parallelogram of forces.

14. Equilibrium of three forces not acting in the same straight line.

Suppose three forces P, Q and R act upon a particle at the point O and keep the particle in equilibrium. The effect of any two out of these three forces, for instance P and Q, would be to move the particle off in a certain direction somewhere between OA and OB, as if it were due to a single force S. But as the particle does not move, the original third force R, must be equal in magnitude and opposite in direction to the combined effect of P and Q, *i. e.*, to S.



It must be noted that P and Q produce the greatest effect possible when they act in the same direction along the same straight line, and this amounts to P+Q. When they are acting along the same straight line but in opposite directions one of the forces directly neutralises the effect of the other, and the combined effect is the least possible, *viz.*, P $\backsim$ Q. When P and Q are not exactly opposite, they are still to a certain extent pulling against each other, and they, therefore, partly destroy each other; consequently the force R has only to overcome the residual effect of the other two forces P and Q, and therefore P+Q is greater than R.

Similarly it can be shown that P+R is greater than Q and R+Q greater than P.

Hence if three forces not acting in the same straight line keep a particle at rest, the *sum* of any two of them must be greater than the third force.

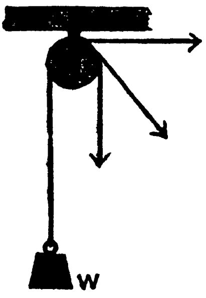
The force S is the *resultant* of P and Q, these latter two are in their turn the *components* of S; while R whose effect is opposite to the joint effects of P and Q, *i.e.*, to that of the resultant S is the *anti-resultant* of P and Q.

The resultant of a system of forces is, therefore, not the same as the force which can keep the system in equilibrium; this latter is the anti-resultant and is equal in magnitude but opposite in direction to the resultant.

Def.—The force which with any given system of forces keeps a particle at rest is called the **anti-resultant** of the system.

Thus, in the figure, the particle O, which would otherwise move under the action of the system of forces composed of P and Q, is kept at rest by R and by definition R is the anti-resultant of P and Q.

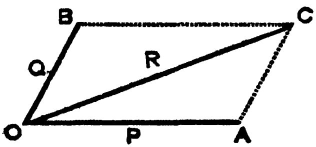
15. **Tension in a string.**—We shall here mention a very important fact determined by careful *experiments.* It is this, that a flexible and inextensible string will transmit force equally well whether it is stretched perfectly straight or is carried over a smooth peg or pulley; for instance, it can be demonstrated that the force required to sustain a weight W is the same, whatever may be the direction of the string after passing over the pulley. The tension between the two equal parts of a string by which a picture frame is suspended is therefore also the same in both the parts.



16. **Parallelogram of Forces.**—The theorem by which the resultant of *any two* forces, acting at any angle

on a particle is determined is called the *Parallelogram of Forces*. It is as follows:—"If **two forces** acting at an "angle on a point be **represented** in magnitude and "direction by the **adjacent sides of a parallelogram**, "the **resultant** of these two forces will be **represented** "in magnitude and direction by the **diagonal** of the pa-"rallelogram passing through this point."

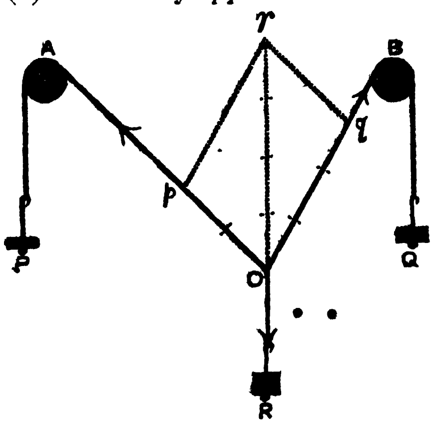
It follows from the above that the resultant is always less than the sum of the two forces.



For if OA and OB represent the two forces and OC the diagonal be the resultant, then OB is equal to AC, and therefore OA plus AC *i.e.*, OA plus OB is greater than OC. (Eu., I, 20). In other words $P + Q > R$, *i. e.*, $R < P + Q$.

Experimental Proof to determine the resultant of two given forces acting at a given angle and thence to demonstrate the truth of the principle of parallelogram of forces.

(*a*) *Details of apparatus.*—Fix two pulleys A and B to a wall, next take three thin cords and knot them at a point O; pass two of the cords round the pulleys and to their free ends attach weights P and Q. Now attach to the third cord successive weights



until the angle between the first two strings is equal to the given angle. When this happens and the three weights are at rest, R is evidently the anti-resultant of P and Q acting at the given angle, and its magnitude will be found to be less than the sum of P and Q; this will be so in every case. *The resultant of P and Q is, therefore, also* equal to R in the same vertical straight line as R, but acting in the opposite direction and is always less than the sum of P and Q.

(*b*) *Construction of the Parallelogram.*—Along the cords OA, OB measure off lengths Op, Oq respectively containing as many units of length as P and Q contain units of mass and on the wall complete the parallelogram $Oprq$ and join Or.

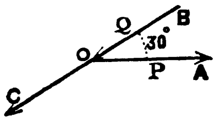
(*c*) *Facts observed.*—Then it will be *always* found—(i) that the diagonal Or is vertical, and (ii) that it contains as many units of length as R contain units of mass, in other words that Or is proportional to R.

(*d*) *Conclusion drawn from the observed facts.*—As Or is in the same line as the anti-resultant but in opposite direction and as it contains as many units of length as the anti-resultant, therefore it represents the resultant both in direction and magnitude, while Op and Oq represent the two forces P and Q.

Hence *the diagonal Or of the parallelogram Oprq represents the resultant of two forces P and Q acting at a given angle, these forces being represented by the sides Op and Oq respectively.*

17. Composition of Forces.—Two forces acting at an angle can thus be reduced to one by the help of the

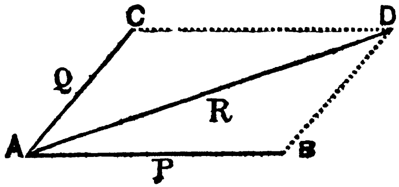
'parallelogram of forces.' We assume that the direction of both forces is either towards or away from the point of application; *if, however, one force be acting towards the point and the other away from it*, then we must replace one of the forces by an equal force acting similarly to the other and next construct the parallelogram. For instance, let forces P and Q be acting on a point O at angle of 30°, P away from the point O and Q towards it; then replace the force *Q* by another equal force acting in the same straight line as Q but away from the point O ('principle of transmissibility') and the parallelogram should be constructed on OA and OC.



If the forces acting be more than two, then we must divide the forces into sets of two each, and by the 'parallelogram of forces' find the resultant of each set, and then taking two of these resultants at a time, find their resultant and follow out the procedure until all the forces are finally reduced to two, and then lastly find the resultant of these two forces; this last, will be the resultant of the original forces. For instance, let there be six forces A, B, C, D, E and F acting on a point and let the resultants of A and B, of C and D and of E and F be respectively P, Q and R; then find the resultant of P and Q, let it be W and lastly find the resultant of W and R, which will be the resultant of the original six forces. If the forces be only five and not six, then the resultant of P and Q, *viz.*, W, should be compounded with E.

18. Half the parallelogram is sufficient to

represent in magnitude and direction the two forces and their resultant.



Let AB and AC represent the two forces P and Q then AD represents the resultant R.

Now as BD is equal and parallel to AC, it also represents Q in magnitude and direction, though not as regards the point of application.

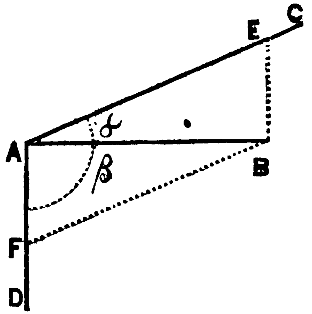
Hence, if two forces (such as P and Q), acting on a particle are represented *only as regards* magnitude and direction by the two sides AB and BD, then their resultant is also represented in magnitude and direction by AD and these three together form half the parallelogram.

Corollary.—It follows, therefore, that the *magnitudes* of any two forces and their anti-resultant, or in other words the magnitudes of any three forces in equilibrium, are always such that the sum of *any two* is greater than the third ; because three forces in equilibrium can always be represented by the sides of a triangle and by geometry two sides of a triangle are always greater than the third.

19. Resolution of forces.—By the 'parallelogram of forces,' just as two forces can be replaced by a single force, so any single force in its turn can be split up or resolved into two definite parts or components. The resolved parts may be at definite angles or not.

Case I.—When the directions along which the force is to be resolved are given.

Suppose it is required to resolve a force P into two components making angles a and b respectively with it on opposite sides.



Construction,—Draw a line AB to represent the force P in all respects. From A draw a line AC making angle BAC equal to a and on the opposite side draw AD making angle BAD equal to b with AB. From B draw BE parallel to AD meeting AC in E and BF parallel to AC meeting AD in F.

Deduction.—Then by the 'parallelogram of forces,' AB is the resultant and AE and AF the components, and as the latter are along the lines AC and AD making angles a and b respectively with the given force P, they are the required components.

Rule.—Hence to resolve a force into any two components along given directions, the rule is to construct a parallelogram whose diagonal is the given force and whose adjacent sides lie along the given directions.

An important case of the resolution of forces is when the force is to be resolved into two components at right angles to one another.—In this case the parallelogram is evidently a rectangle. The components are called the *resolutes* or

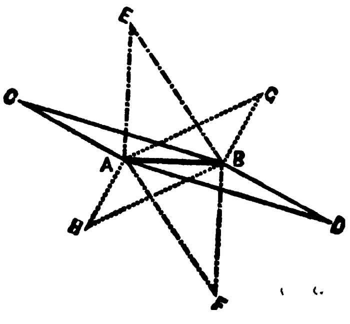
resolved parts of the original force along their respective lines of action.

Def—When a force is resolved into two components at right angles to each other, each component is called the **resolute** or **resolved part** of that force in the given direction.

In the case of resolving a force into two components at right angles to one another, we must note that *each of the resolutes is always less than the force resolved*, for the original resolved force always forms the diagonal of a rectangle, *i. e.*, the hypotenuse of a right-angled triangle and the resolutes form the sides containing the right angle; now as the hypotenuse is the greatest side in a right-angled triangle, the original force represented by the hypotenuse is greater than either of the resolutes represented by the other sides of the triangle.

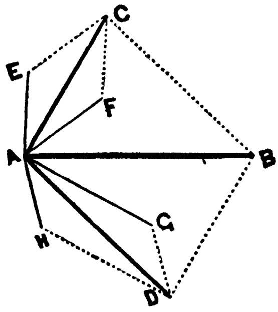
Case II.—When the directions along which the force is to be resolved are not given.

In this case a force may be resolved into two components in an indefinite number of ways, because the same line may be the diagonal of an infinite number of parallelograms.



For instance, in the figure, AB is at one and the same time the diagonal of several parallelograms, such as ACBD, AEBF, AGBH, and the components are AC and AD, AE and AF and AG and AH, respectively.

Case III.—Resolution of a force into more components than two.

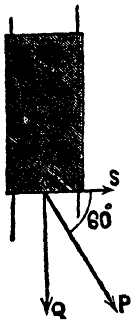


If a force AB be first resolved into two forces AC and AD, each of these can again be resolved into two, such as AE, AF and AH, AG ; this may be repeated to any extent and thus a single force may be resolved into any number of forces at a time.

20. Effect of a force in a direction inclined to its own line of action.

A force cannot produce any effect in a direction perpendicular to its own line of action. For example, if a tram-car is standing at rest on the rails, no amount of force applied at *right angles* to the rails will be of any use in moving the car along the rails, as the only tendency of such a force will be to press the wheels against the grooves in the rails. On the other hand, if the car is pulled or pushed *parallel* to or along the rails, then no part of the force will be spent in producing pressure between the wheels and the rails, and the force will have its full effect in causing motion.

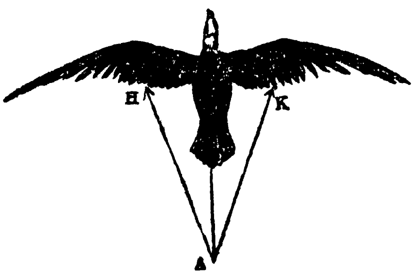
If, however, the force is acting at an angle to the rails, as, for instance, when the horse swerves to one side, then only a portion of the whole force becomes available. In the figure, P is supposed to be applied at an angle of 30° with the car, and we must replace P by its resolutes Q and S, one along the direction of motion and the other at right angles to it. S produces pressure between the car and the rails and is useless for our purpose; it is Q alone which causes motion along the rails.



As the resolute of a force is always less than the force itself (*vide* article 19), it is clear that the effect of a force is reduced when its direction is inclined to the direction of motion of the body.

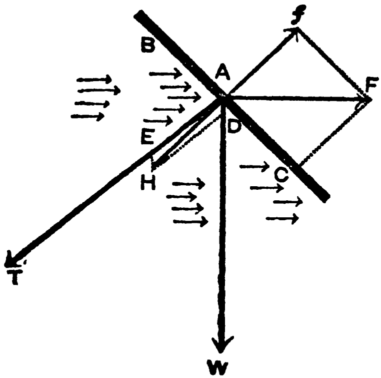
21. Examples illustrating the composition and resolution of forces.

FLIGHT OF BIRDS.—A bird propels itself by two equal forces generated by flapping the wings towards its tail. Thus AH and AK respectively indicate the directions in which the bird is propelled by each of the wings; these two forces meet at A, and by the 'composition of forces,' the bird flies in the direction of the diagonal of the parallelogram formed on AH and AK as two adjacent sides.



We may vary the illustration by making one force greater than the other, than by reconstructing the figure it will be seen at once that the direction of the resultant force, and therefore the direction in which the bird will fly, will lean towards the greater force.

A KITE IN MID-AIR—It is the force of the wind which makes the kite fly. The wind strikes against the under-surface of the kite, but not always at right angles to it. Generally therefore, the force of the wind may be resolved into two components : one along the surface of the kite, which will produce no effect and the other at right angles to it which will propel the kite forward.



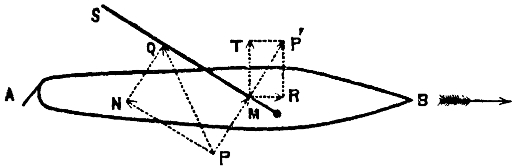
In the accompanying diagrammatic sketch, BC is the kite seen edge-wise, the small arrows indicate the direction of the wind, AF is the total force of the wind, and A*f* the effective component which helps the kite forward in its flight.

But when the kite remains hanging in mid-air, there must be other forces acting against the force A*f*. These are evidently two, *viz.*, the weight of the kite acting vertically downwards and the pull or tension which is always felt on the string, both acting against A*f* ; let them be represented by AD and AE, respectively, then as the kite remains stationary their resultant AH must of necessity be equal and opposite to A*f*.

To recapitulate, there are three forces always acting on a kite, *viz.*, the wind-component at right angles to the

surface of the kite, the weight of the kite and the tension on the string; if the first be greater than the resultant of the last two, the kite flies onward; when equal it remains hanging in mid-air; but when it is less then the kite falls back to the ground.

A SHIP SAILING AGAINST THE WIND.—Let AB represent the plan of a ship, MS its sail, and PQ the relative direction and strength of the wind.



From P drop perpendicular PM on MS and complete the parallelogram PMQN.

Then the force of the wind denoted by PQ may be replaced by the two forces PN and PM. We may, therefore, consider the original wind to be replaced by two winds, one blowing parallel to or along the surface of MS and the other perpendicular to it.

It is obvious that the first of these winds can have no effect, as it merely runs parallel to the sail without pressing upon it. We may, therefore, discard this component and consider only the wind PM, which blows perpendicularly upon the sail.

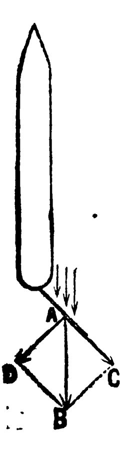
On the principle of 'transmissibility of forces,' produce PM and make MP′ equal to PM, and resolve it along the length of the ship and at right angles to it, *i.e.*, into MR and MT. The force, therefore, which produces *the desired*

effect, i.e., the motion of the ship in a line with the length of the ship, is represented by the component MR ; while the effect of the other component MT, which tends to move the ship in a direction perpendicular to the length of the ship, is counterbalanced by the resistance of the water against the side of the ship.

The shape of the ship is so designed that the resistance of the water in the direction MR in which it sails is very small ; whilst, on the other hand, it encounters very great resistance from the water, when the component MT tends to move it sideways. Thus, though no force can exert pressure exactly at right angles to itself, yet it is easy for a ship to sail at right angles to the wind, because though the ship may be at right angles to the wind, the sails are not ; they act as a mediary, being inclined to both wind and ship.

Note that the sail must always be set between the directions of the motion and of the wind or else the ship cannot move forward.

ACTION OF SHIP'S RUDDER.—As the ship passes through the water, the water offers resistance and presses against the rudder. This is indicated in the figure by the small arrows.

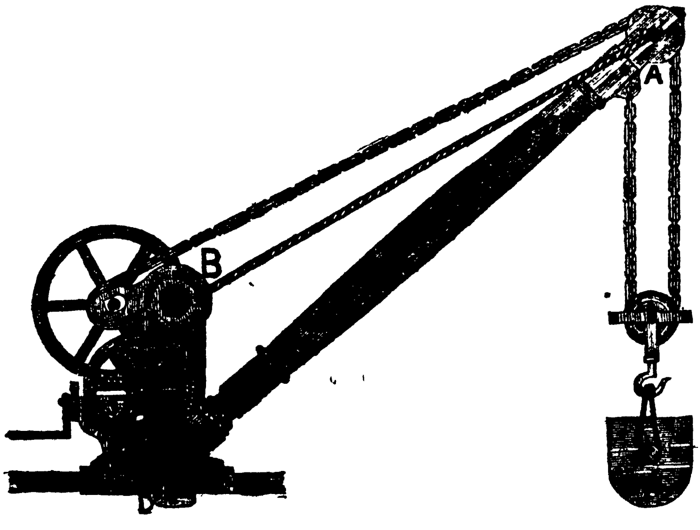


Let the total effect of this opposed force be represented by AB, then this force can be resolved into two forces, AC and AD, along the surface of the rudder and at right angles to it respectively. The former can produce no effect ; the latter, however, will turn the rudder (in the figure, from right to left), and the ship's head will, therefore, be turned in the opposite direction, from left to right.

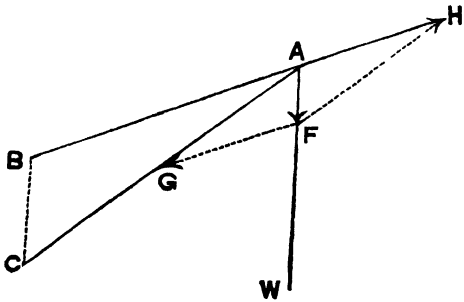
THE CRANE OR JIB-AND-TIE.—In unloading heavy weights from railway waggons or from ships at the docks, it is necessary first of all to lift up the weight and next to carry it safely from the waggon or the ship to the platform or the wharf. A machine intended for this purpose must, therefore, be able to perform three distinct operations :—in the first place it must be a sufficient mechanical power to lift up large weights, in the second place it must be sufficiently strong to hold safely the load suspended from it, and in the third place it must be capable of moving the load away.

The first requirement is satisfied by the hoisting apparatus, which is a combination of a winch (or wheel and axle) and a moveable pulley, while the last and third operation is performed by the whole frame-work being made to rotate on a strong pivot shown at D.

We will next consider how the second requirement is fulfilled, and how the various parts of the frame-work are held *in equilibrium* with the weight hanging from the top of the crane.



The beam AC is called the **jib**, and the strong cord AB (which is often replaced by a chain or a rod) is called the **tie.** The weight W acts vertically downwards in the direction AW. Mark off AF to represent the magnitude of the weight, and through F draw FG parallel to the tie AB, meeting AC in G. On GA and GF as adjacent sides describe the parallelogram FGAH, then AF will be the diagonal of the parallelogram and *the forces AG and AH will keep the weight (W) in equilibrium.* The result of the weight W hanging from A, therefore, is that it exerts a **thrust** downwards or a force of compression on the jib, and a **strain** outwards or a force of extension on the tie. The former is representnd in magnitude and direction by AG and the latter by AH.



THE TOWING OF A BOAT along the banks of a canal or narrow river is another instance of the composition of forces.

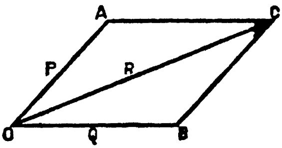
Here two equal forces are applied in the directions AB and AC and therefore the boat is pulled forwards in the direction AE equally inclined to AC and AB, which is that of the resultant of the two forces.

In the article that follows it is proved that the resultant varies inversely as the angle between the components. Therefore, if the towing-ropes be long, the angle between them is small and with the same exertion a greater effect is produced on the boat.

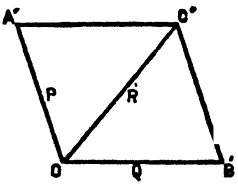
22. Magnitude and direction of the resultant.

We will now proceed to show how the magnitude and the direction of the resultant changes with the magnitude of the components and the included angle.

(*a*) *Magnitude of the resultant varies inversely as the angle between the forces.*



Let P and Q be two forces and let the angle between them be first AOB and secondly A′OB′, which is greater than AOB.



Let the two forces P and Q be represented by OA and OB in the first case, and by OA′ and OB′ in the second case. Complete the parallelograms AOBC and A′OB′C′, then OC and OC′ are the respective resultants R and R′.

Now angles AOB and OBC are together equal to two right angles (Eu. I, 29), so are also angles A′OB′ and OB′C′; therefore angle AOB+angle OBC=angle A′OB+angle OB′C′.

But angle AOB is less than angle A′OB′

∴ angle OBC is greater than angle OB′C′,

also BC=B′C′, as both are equal to P,

therefore, in the triangles OBC and OB′C′,

the sides OB and BC are respectively equal to OB′ and B′C′,

but the contained angle OBC is greater than the contained angle OB′C′,

∴ the side OC is greater than the side OC′ (Eu. I, 24)
i.e., R. is greater than R′.

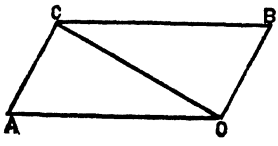
Hence *the greater the angle between two forces the less is the resultant and vice versa.* Q.E.D.

It may be remarked that the greatest angle between two forces is 180°, the forces then act in exact opposition to one another and the resultant then is the least possible (cf. article 11).

(*b*) *The resultant is nearer the greater force.*

Let OA add OB represent two forces P and Q acting on the point O, of which OA is the greater. Complete the parallelogram AOBC and join OC.

Then OC, the resultant, will be nearer to OA than to OB.



As OA = BC and OA is greater than OB;

∴ BC is greater than OB, and the angle COB is greater than the angle OCB (Eu. I., 18.)

But angle OCB = angle COA,

∴ angle COA is less than angle COB,

and OC is nearer OA than OB ;

i.e., the resultant is nearer the greater force.

Q.E.D.

Cor. If the two forces be equal, then the resultant will be equally inclined to the two forces.

Because then OA = BC = OB,

∴ angle BOC = angle BCO (Eu. I., 5.) and angle BCO = angle AOC (Eu. I., 29.),

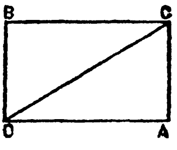
∴ angle BOC = angle AOC

and OC is equally inclined to OA and OB. Q.E.D.

23. Formulæ for the resultant of two forces acting at a point at any angle.

(1) The forces are at right angles.

Let OA and OB represent two forces P and Q, and let the angle AOB be a right angle.



Complete the rectangular parallelogram AOBC and join OC. Then, by the 'parallelogram of forces,' OC will represent R, the resultant of P and Q.

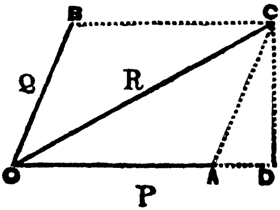
Now, since the angle OAC is also a right angle,

$$OC^2 = OA^2 + AC^2 \text{ (Eu. I., 47)}$$
$$= OA^2 + OB^2,$$
$$\therefore R^2 = P^2 + Q^2,$$
$$\text{and } \mathbf{R} = \sqrt{P^2 + Q^2}.$$

Q. E. D.

(2) General expression when the forces act at an angle less than a right angle.

Let OA and OB represent the two forces P and Q acting at the point O, and let the angle AOB be less than a right angle.



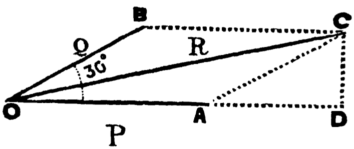
Complete the parallelogram AOBC and join OC; then by the 'parallelogram of forces,' OC represents the resultant.

From C draw CD perpendicular to OA produced, then

$$OC^2 = OA^2 + AC^2 + 2\,OA.AD \quad \text{(Eu. II., 12.)}$$
$$= OA^2 + OB^2 + 2\,OA.AD,$$
$$\therefore \mathbf{R^2 = P^2 + Q^2 + 2\,P.AD.}$$

The more **important special cases** involving this general formulæ are those of angles of **30°**, **45°** and **60°**.

(a) Let the angle AOB be **30°**;



then the angle CAD $= 30°$, (Eu. I., 29)
and the angle ACD $= 60°$; (Eu. I., 32)
and therefore CD $= \frac{1}{2}$AC. (*vide* Appendix B.)

Now, since the angle ADC is a right angle,

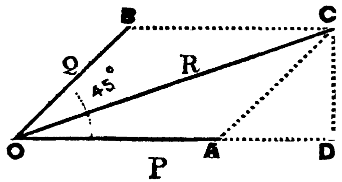
$$AC^2 = CD^2 + AD^2$$
$$= (\tfrac{1}{2}\,AC)^2 + AD^2,$$
$$\therefore AD^2 = \tfrac{3}{4}\,AC^2$$
$$\text{and } AD = \frac{\sqrt{3}}{2}AC\,;$$

but $AC = OB = Q$, $\therefore AD = \frac{\sqrt{3}}{2} Q$,

and by substitution in the general formulæ,

$$\mathbf{R^2 = P^2 + Q^2 + P.Q\sqrt{3}}.$$

(b) Let the angle AOB be **45°**;
then the angle CAD = 45°,
and the angle ACD = 45°,
and triangle ACD is isosceles.



Now since the angle ADC is a right angle,

$$AC^2 = AD^2 + CD^2,$$

$$\text{but as } AD = CD$$

$$AC^2 = 2\,AD^2,$$

$$\text{and } \tfrac{1}{2}\,AC^2 = AD^2\,;$$

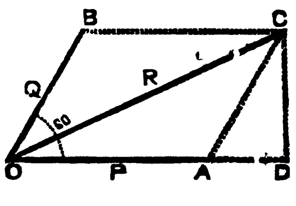
$$\therefore AD = \frac{1}{\sqrt{2}} AC,$$

$$= \frac{1}{\sqrt{2}} OB = \frac{1}{\sqrt{2}} Q\,;$$

and by substitution in the general formulæ,

$$\mathbf{R^2 = P^2 + Q^2 + P.Q\sqrt{2}}.$$

(c) Let the angle AOB be **60°**;



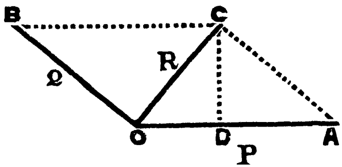
then the angel CAD = 60°,
and the angle ACD = 30°,
and $AD = \frac{1}{2} AC = \frac{1}{2} OB = \frac{1}{2} Q$;
∴ by substitution in the general formulæ,

$$R^2 = P^2 + Q^2 + P.Q.$$

(3) General expression when the forces act at an angle greater than a right angle.

Let OA and OB represent the two forces P and Q, acting at the point O, and let the angle AOB be greater than a right angle.

Complete the parallelogram AOBC and join OC. Then by the 'parallelogram of forces,' OC represents the resultant.



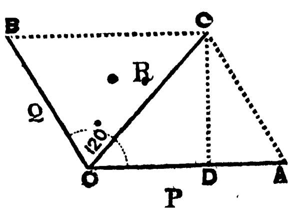
From C draw CD perpendicular to OA, meeting OA in the point D.

Then by Euclid, Bk. II., Prop. 13,

$$OC^2 = OA^2 + AC^2 - 2\ OA.AD,$$
$$= OA^2 + OB^2 - 2\ OA.AD,$$
$$\therefore R^2 = P^2 + Q^2 - 2\ P.\ AD.$$

The more **important special cases** involving this general expression are those of angles of **120°**, **135°**, and **150°**.

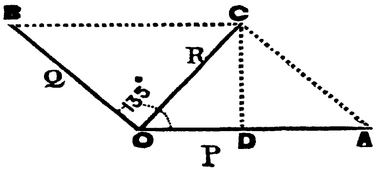
(a) Let the angle AOB = **120°**,



then the angle CAD $= 60°$,
and consequently the angle ACD $= 30°$;
then AD $= \frac{1}{2}$ AC $= \frac{1}{2}$ OB $= \frac{1}{2}$ Q,
and by substitution

$$R^2 = P^2 + Q^2 - P.Q.$$

(b) Let the angle AOB $= 135°$,



then the angle CAD $= 45°$,
and the angle ACD $= 45°$,
and the triangle ACD is isosceles.

Now since the angle ADC is a right angle,

$$AC^2 = AD^2 + CD^2, \text{ (Eu. I. 47)}$$

but as AD $=$ CD,

$$AC^2 = 2\ AD^2,$$

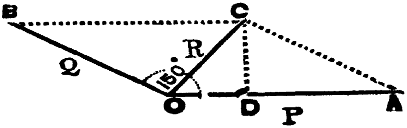
and $\frac{1}{2}\ AC^2 = AD^2$,

$$\therefore AD = \frac{1}{\sqrt{2}} AC = \frac{1}{\sqrt{2}} OB = \frac{1}{\sqrt{2}} Q;$$

and by substitution

$$R^2 = P^2 + Q^2 - P.Q.\sqrt{2}.$$

(c) Let the angle AOB $= 150°$,



then the angle CAD $= 30°$,
and the angle ACD $= 60°$,
and therefore CD $= \frac{1}{2}$ AC.

Now since the angle ADC is a right angle,

$$AC^2 = CD^2 + AD^2,$$
$$= (\tfrac{1}{2}\,AC)^2 + AD^2,$$
$$= \tfrac{1}{4}\,AC^2 + AD^2,$$
$$\therefore AD^2 = \tfrac{3}{4}\,AC^2,$$
$$\text{and } AD = \frac{\sqrt{3}}{2}\,AC;$$
$$\text{but } AC = OB = Q,$$
$$\therefore\ AD = \frac{\sqrt{3}}{2}\,Q;$$

and by substitution

$$\mathbf{R^2 = P^2 + Q^2 - P.Q.\sqrt{3}}.$$

Summary of results :—

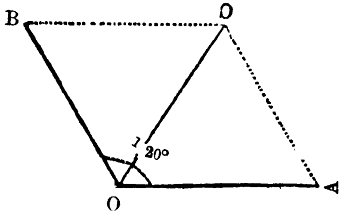
For 0°, $R^2 = P^2 + Q^2 + P.Q.\sqrt{4}$.
,, 30°, $R^2 = P^2 + Q^2 + P.Q.\sqrt{3}$.
,, 45°, $R^2 = P^2 + Q^2 + P.Q.\sqrt{2}$.
,, 60°, $R^2 = P^2 + Q^2 + P.Q.\sqrt{1}$.
,, 90°, $R^2 = P^2 + Q^2 + P.Q.\sqrt{0}$.
,, 120°, $R^2 = P^2 + Q^2 + P.Q.\sqrt{1}$.
,, 135°, $R^2 = P^2 + Q^2 - P.Q\ \sqrt{2}$.
,, 150°, $R^2 = P^2 + Q^2 - P.Q.\ \sqrt{3}$.
,, 180°, $R^2 = P^2 + Q^2 - P.Q.\sqrt{4}$.

Comparing these results, we see that the value of R^2 and therefore of R is greatest when the angle between the forces is zero, *i.e.*, when the forces act in the same direction along the same straight line and it goes on diminishing as the value of the angle increases, until it reaches a minimum when the angle becomes 180°, *i.e.*, when the forces act in the same straight line but in opposite directions.

24. Equal forces acting at an angle of 120°.

This case deserves separate consideration.

(*a*) *Let OA and OB represent two equal forces* **P** *and Q acting on a particle O at an angle of* 120°, *then their resultant is equal to either of them.*



Complete the parallelogram OACB and draw the diagonal OC ; then by the 'parallelogram of forces' OC will represent the resultant R in magnitude and direction.

Now since

$$AC = OB, \quad \text{(Eu. I., 34)}$$

$$AO = AC,$$

$$\text{and angle } AOC = \text{angle } ACO, \quad \text{(Eu. I., 5)}$$

$$\text{also angle } BOC = \text{angle } ACO, \quad \text{(Eu. I , 29)}$$

$$\therefore \text{ angle } AOC = \text{angle } BOC,$$

$$\textit{i.e.}, \text{ angle } AOC = \tfrac{1}{2} \text{ whole angle } AOB = 60°,$$

$$\text{also angle } ACO = 60°,$$

$$\text{and therefore also angle } CAO = 60° \text{ (Eu. I., 32)}$$

and the triangle ACO is equilateral.

Hence OC the resultant is equal to either of the equal forces OA or OB.

(*b*) *Conversely, if the resultant of two equal forces be equal to each of them, the angle between them is* 120.°

For let OA and OB be the two equal forces and let their resultant OC be also equal to each of them.

Then by the 'parallelogram of forces,' OC is the diagonal of the parallelogram OACB,

and OB = AC, (Eu. I. 34)

∴ triangle AOC is equilateral,

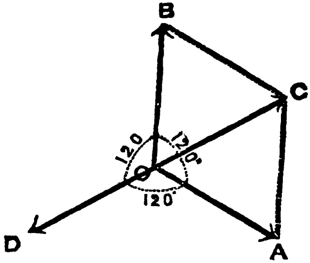
and angle AOC = 60° ; (Eu. I., 32)

similarly angle BOC = 60°,

∴ angle AOB = 120°.

(*c*) *If three equal forces keep a particle in equilibrium, they are equally inclined to one another and conversely if three equally inclined forces keep a particle in equilibrium, the forces are all equal.*

Let OA, OB, and OD be three *equal* forces keeping the particle O in equilibrium, then they are equally inclined to one another.



Since any one of them OD is the anti-resultant of the other two OA and OB (*vide* article 14), therefore OC, the resultant of OA and OB, is also equal in magnitude to OD, *i.e.*, to either OA, OB ; hence the angle AOB is 120°. (*Vide supra, Case b.*)

Now as OC is in the same line as OD,

therefore, angle COB + angle BOD = 180°,

and angle BOC = 60°, (*vide supra, Case b*)

therefore, angle BOD = 120° ;

similarly angle AOD = 120°.

Each of the angles AOB, BOD and DOA is thus equal to 120°, in other words the three equal forces OA, OB and OD are equally inclined to one another.

Conversely if OA, OB and OD are in equilibrium and equally inclined to one another they will be all equal.

Because as the three forces are in equilibrium, OD is the anti-resultant of OA and OB and the resultant OC is equal to and in the same straight line as OD.

But as angle BOD = 120°,

∴ angle BOC = 60°,

similarly angle COA = 60°;

therefore angle OCB = 60°, (Eu. I., 29.)

and also angle OBC = 60°,

hence triangle OCB is equilateral, and OC = OB.

In the same manner it can be shown that OC = OA,

∴ OC is equal to both OA and OB,

but OC is equal to OD,

∴ OD is equal to both OA and OB;

i.e., OA, OB and OD are all equal.

25. Graphical Method of Solution.—Every case of composition or resolution of forces acting at an angle involves the solution of a parallelogram or a triangle, and in every case either the three forces are given or two of them and an angle are given, therefore if we construct a parallelogram or triangle in conformity with these data by the help of scale and protractor, the figure will be correct in every respect and the requisite information can be obtained by inspection.

QUESTIONS.

1. Without taking the help of the 'parallelogram of forces,' show that when three forces not acting in a straight line equilibriate at a point, the sum of any two of them is greater than the third.

2. Enunciate the proposition known as the '*parallelogram of forces*' and describe an experiment whereby it may be verified.

3. If in the above experiment you begin with three definite weights, how will you proceed to prove the proposition ?

4. Distinguish between *resultant and anti-resultant.* Does the third weight in the experiment in Ques. 2 directly give the resultant or the *anti-resultant* of the other two ?

5. What do you mean by *resolving* a force into its components and what by *compounding* a number of forces ?

6. What is the difference between *resolutes* and *components* of a force ?

7. Show that resolutes are always less than the original force, and hence indicate why it is most advantageous to apply a force in the direction in which the effect is to be produced.

Does the above proposition hold good in the case of components at *any* angle ?

8. Show why the traces of a horse ought always to be parallel to the road along which he is pulling.

9. Show that two forces and their resultant can always be represented by the three sides of a triangle in *magnitude* and *line of action* but not as regards their *direction*.

10. How do the sails of a ship help her to travel forward irrespective of the direction of the wind except when it is blowing a 'head-wind ?'

Why do the sails fail in the last case ?

11. How does the rudder help a ship to change her course ?

12. Describe the *crane.* Why is the *jib* always of rigid wood or metal and not the *tie* ?

13. Explain the flight of birds on the principle of the parallelogram of forces.

14. What forces are acting on a kite when it remains stationary in mid-air ?

15. How is a boat towed along a canal by the help of ropes?

16. Show that the resultant is always nearer the greater force.

17. How does the resultant vary with the angle between the forces? Obtain the result by calculation and also graphically.

18. Obtain expressions for the resultant of two forces when acting—

(1) at right angles,

(2) at less than a right angle,

(3) at 30°, 45°, 60°,

(4) at more than a right angle,

(5) at 120°, 135°, 150° and 180°,

and tabulate the results in ascending order.

19. Give an independent proof of the fact that two *equal* forces acting at an angle of 120° have a resultant equal to either of them.

EXAMPLES.

1. Prove that the forces are equal, if the resultant bisects the angle between them.

2. If a system of forces be in equilibrium, prove that each of these forces is equal to the resultant of all the rest, and acts in a direction directly opposite to the direction of that resultant.

3. Prove that the resultant is equal to twice the line joining the point of action with the middle of the line which joins the free extremities of the lines representing the two forces,

(*N. B.*—The diagonals of a parallelogrem bisect each other.)

4. Can forces represented by 2, 3, and 6 lbs. be in equilibrium? Give the reason.

No.

Because if these three forces be in equilibrium, one of them in intensity must be equal to the resultant of the other two. And as by the 'parallelogram of forces,' two forces and their resultant must be as the sides of a triangle (*vide* Article 18), lines in proportion of 2, 3, and 6 must form such a triangle,

But (by Euclid, Bk. I., Prop. 20) any two sides of a triangle are greater than the third,

∴ *any* of the two forces must be greater than the third.

∴ 2 + 3 must be greater than 6, which is absurd.

5. If equal forces are added to two forces P and Q acting at an angle, determine the change thereby produced in the *direction* of R.

6. Why is no similar change produced when the original forces are increased or diminished in equal proportions?

7. When a bullock is employed to tow a boat along a canal, the tow-rope is usually of considerable length: give a definite reason for using a long rope instead of a short one.

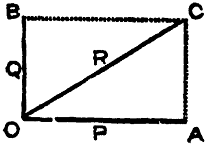
8. A heavy plummet is immersed in a stream, the string being held by a person standing on the bank, when the string is found to settle in a sloping position. Show by means of a sketch the three forces which keep the plummet in equilibrium.

9. Three forces are represented by the lines joining an angle of a parallelogram to the other angles. Find a line representing their resultant and its magnitude. *Ans.* If OABC be the parallelogram and the forces be represented by OA, OB, OC, then the resultant is 2 OB, both in magnitude and direction.

10. A boat is steered due north with the same force as the current itself, which flows from east to west: what is the direction taken by the boat? *Ans.* North-west.

11. Two forces represented by 10 lbs. and 15 lbs. respectively act on a point and make with one another an angle of 90°. Find the magnitude of the resultant.

Let OA, OB represent the two forces of 15 lbs. and 10 lbs. respectively.



Complete the parallelogram AOBC and join OC, then OC will represent the resultant.

Because angle AOB is a right angle and OA is parallel to BC,

∴ angle OBC is also a right angle. (Eu. I., 29.)

Now in the right-angled triangle OBC

$$\therefore \quad OC^2 = OB^2, + BC^2, \quad \text{(Eu. I., 47)}$$

$$\text{but } BC = OA,$$

$$\therefore OC^2 = OB^2 + OA^2,$$

$$\therefore OC^2 = 10^2 + 15^2,$$

$$= 325,$$

$$\text{and } OC = \sqrt{325} = 18{\cdot}02 \text{ nearly;}$$

$\therefore$ the resultant is one of 18·02 lbs. nearly.

Alternative proof :—

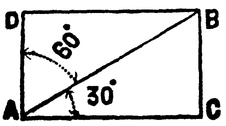
$$\text{By article 23, } R^2 = P^2 + Q^2,$$

$$= 10^2 + 15^2,$$

$$\therefore R = \sqrt{325} = 18{\cdot}02 \text{ nearly.}$$

12. Find the intensities of the two components of a force resolved at angles of 30° and 60° respectively with the original force.

Let R, the original force, be represented by AB and the component P making the angle of 30° with R by the line AC, and the other component Q at 60° with R by AD; then, by 'the parallelogram of forces,' ACBD will be a right-angled triangle.



By geometry, in the right-angled triangle ACB, as AC is the side next the angle of 30°, therefore, $AC = \frac{\sqrt{3}}{2} AB$; (*vide* Appendix B) and again as in right-angled triangle ADB, AD is the side next the angle of 60°,

$$AD = \frac{1}{2} AB.$$

$$\text{Hence } P = \frac{\sqrt{3}}{2} R,$$

$$\text{and } Q = \frac{1}{2} R.$$

13. Two forces act upon a point at right angles to one another; the resultant is 51 lbs. and one component 24 lbs., find the other.

By article 1?, the two forces and their resultant must form a right-angled triangle.

Let P denote the unknown force in pounds, then $R^2 = P^2 + Q^2$,

$\therefore 51^2 = P^2 + 24^2$ and $P = 45$ lbs.

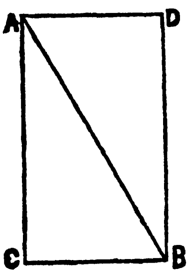
14. The resolute of a force making an angle of 60° with it is 6 lbs. what is the force and what is the value of the other component? (*Vide* figure of example 12.)

Let AB be the original force, AD the resolute at 60°, then the other resolute AC will make angle of 30° with AB.

By hypothesis AD is equal to 6, and therefore in the triangle ABC, BC is equal to 6 and angle ABC is 60°, angle BOC is 30° and angle ACB is 90°;

$$
\begin{aligned}
\text{Hence } AB &= 2\ BC, \text{ (Appendix B.)} \\
&= 12 \text{ lbs.,} \\
\text{and } AC &= \sqrt{OA^2 - AC^2} \\
&= \sqrt{144 - 36} \\
&= \sqrt{108} \\
&= 6\sqrt{3} \text{ lbs.}
\end{aligned}
$$

15. A particle is acted on by a force whose magnitude is unknown, but whose direction makes an angle of 60° with the horizon. The horizontal component of the force is known to be 1·35 units, determine the total force and also its vertical component.



Let AD be the horizontal component, and let AB be the *direction* of the total force. From D draw DB at right angles to AD meeting AB in B, and complete the parallelogram ADBC; then by the 'parallelogram of forces,' AB represents the total force, both in direction and intensity, and AC is the vertical component.

As the angle BAD is 60° and angle BAC = 30°,

$\therefore \frac{1}{2}$ AB = AD = 1·35,

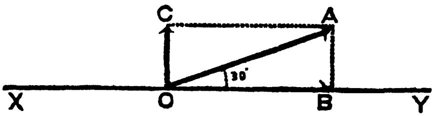
and AB = 2·7 units.

Again, as $BD = \frac{\sqrt{3}}{2} AB$

and AB = 2·7 and AC = BD,

$\therefore$ AC = 2·34 nearly.

16. A man pulls a weight by means of a rope along a road with a force of 20 lbs. The rope makes an angle of 30° with the road, find the force he would need to apply parallel to the road.



Let the weight O be pulled along the road X Y from left to right by the force O A, then O A is equal to 20 lbs. and is inclined at an angle of 30° to O Y.

The *effectual* force which moves the weight is the resolute of O A along X Y (*vide* article 20) and a force equal to this resolute applied along X Y would produce the same effect.

From A drop perpendicular AB on OY and complete the right-angled parallelogram OBAC, then OB is the resolute required.

Now angle A O Y = 30°,

and angle O B A = 90°,

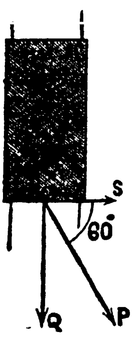
∴ angle O A B = 60°,

and side OB opposite to angle 60° $= \frac{\sqrt{3}}{2}$ O A, (*Vide* Appendix. B.)

$= \frac{\sqrt{3}}{2} \times 20.$

$= 10\sqrt{3} = 17{\cdot}3$ lbs. nearly.

17. A truck is at rest on a railway line and is pulled by a rope with a horizontal force equal to the weight of 100 lbs. in a direction making an angle of 30 with the direction of the rails; what is the force tending to urge the truck forwards?



Let P represent the force of 100 lbs. making the angle of 30° with the direction of the rails.

Resolve P along the direction of the rails and at right angles to it. Then the resolute Q is the *effectual* component moving the truck along the line; find Q.

Completing the triangle formed by P and Q, we find that Q is opposite the angle of 60°,

$$\text{hence } Q = \frac{\sqrt{3}}{2} P, \quad (\textit{Vide} \text{ Appendix B})$$

$$= \frac{\sqrt{3}}{2} \times 100$$

$$= 50\sqrt{3} \text{ lbs.}$$

18. At what angle must forces of 6 lbs. and 8 lbs. be acting so as to be kept in equilibrium by a force of 10 lbs.

(*Vide* summary at the end of Article 23.)

$$R^2 = P^2 + Q^2 + P.Q.\sqrt{x} \quad \text{(General formula);}$$

by substitution—

$$10^2 = 6^2 + 8^2 + 6 \times 8\sqrt{x}.$$

$$100 = 36 + 64 + 48\sqrt{x}.$$

$$\therefore 48\sqrt{x} = 0$$

$$\text{and } \sqrt{x} = 0$$

and the formula, therefore, corresponds with 90°.

19. If forces of 3 grammes and 4 grammes have a resultant of 5 grammes, at what angle do they act? *Ans.* 90°.

$$R^2 = P^2 + Q^2 + P.Q\sqrt{x}.$$

By substitution—

$$5^2 = 3^2 + 4^2 + 12\sqrt{x}$$

$$\therefore \sqrt{x} = 0 \quad \therefore \text{the angle is } 90°.$$

20. Find the resultant of the following pairs of forces acting at right angles to each other:—

(1) 70 and 24. (2) 60 and 144, (3) 18 and 80, (4) 3 and 4.
(5) 10 and 20, (6) 24 and 143, (7) 10 and 12, (8) 20 and 21,

Ans.
(1) 74, (5) 22·36, or
(2) 156, (6) 145,
(3) 82, (7) 15.62,
(4) 5, (8) 29.

21. Three cords are tied together at a point. One of these is pulled in a northerly direction with a force of 6 lbs. and another in an easterly direction with a force of 8 lbs. With what force must the third cord by pulled in order to keep the whole at rest? *Ans.* 10 lbs.

22. Find the resultant of three forces of which two are forces of 12 and 15 units acting in opposite directions along the same line, and the third force of 8 units acting along a line at right angles to the former lines. *Ans.* $\sqrt{73}$.

23. Two forces, one of which is three times the other, act along the adjacent sides of a square; find the resultant. *Ans.* $\sqrt{10}$ times the small.

24. Two forces, whose magnitudes are as 3 to 4, act at a point in directions at right angles, and produce a resultant of 2 lbs. Find the forces. *Ans.* $1\frac{1}{5}$ and $1\frac{3}{5}$ lbs.

25. Two forces whose magnitudes are as 3 is to 4, acting on a particle at right angles to each other, produce a result of 15 units: find the forces. *Ans.* 9 and 12.

26. The horizontal and vertical components of a certain force are 5 lbs. and 12 lbs. respectively: what is the magnitude of the force? *Ans.* 13 lbs.

27. Resolve the force 12 into two forces, making angles of 45° with the given force on either side of it. *Ans.* $6\sqrt{2}$.

28. A force of 13 units acts along a line making an angle of 30° with a given a line: find, *by construction*, its components along and at right angles to that line.

29. Four equal forces act on a point; the first is at right angles to the second, the third at right angles to the resultant of the first two, and the fourth at right angles to the resultant of the other three: find the resultant of all four.

For an angle of 90°, $R^2 = P^2 + Q^2$,

$\therefore$ the resultant of the first and the second is equal to

$$\sqrt{2P^2} = P\sqrt{2};$$

and the resultant of this last and the third force is equal to

$$\sqrt{P^2 + 2P^2} = P\sqrt{3};$$

and the final resultant of all four forces is equal to $\sqrt{P^2 + 3P^2} = 2P$.

30. Find the resultant of the following pairs of forces acting upon a point at an angle of 30° :—

(1) 10 and 20, (2) 30 and 18, (3) two forces of 51·76 lbs. each, (4) 20 and 51·96.

Ans. (1) 29·09. (2) 47. (3) 100. (4) 70 nearly.

31. Find the resultant of the following pairs of forces acting at an angle of 45° :—

(1) Two forces each of 10 lbs., (2) 30 and 18, (3) 10 and 20.

Ans. (1) 18·477 lbs., (2) 44, (3) 27 97.

32. Find the resultant of the following pairs of forces acting upon a point at an angle of 60° :—

(1) Two forces of 1C0 each. (2) 36 and 60. (3) 18 and 6. (4) 10 and 20. (5) two forces of 57·735 each. (6) two forces of ?$\sqrt{3}$ lbs. each. (7) two forces of $20\sqrt{3}$ each. (8) two forces of $2\sqrt{3}$ lbs. each.

Ans. { (1) 173·2, (2) 84, (3) 21·3, (4) 26·45, (5) 1C0, (6) 9, (7) 60, (8) 6.

33. Find the resultant of the following pairs of forces actıng upon a point at an angle of 120° :—

(1) 36 and 96, (2) 70 and 400, (3) 60 and 160, (4) two forces of 30 lbs. each.

Ans. (1) 84, (2) 370, (3) 140, (4) 30.

34. Find the resultant of the following pairs of forces acting upon a point at an angle of 135° :—

(1) Two forces of 130·653 lbs. each, (2) 98 and 147, (3) 8 and 9.

Ans. { (1) 100, nearly, (2) 104·125. (3) 6·55.

35. Find the resultant of the following pairs of forces acting upon point at an angle of 150° :—

(1) Two forces of 100 lbs. each, (2) 98 and 147, (3) 25 and 15.

Ans. { (1) 51·7638, (2) 78·89, (3) 14·15.

36. A boat is moored in a stream by two ropes fixed to posts, one on each bank, and inclined to the direction of the current, at angles of 30° and 45°. Draw a figure from measuring which, the proportions may be found between the strains on the ropes.

37. A ship is anchored to two buoys by chains, which make an angle of 60° with each other. The strain on each chain is 2,000 lbs. Find the force of the stream. *Ans.* $2000\sqrt{3}$ lbs.

38. A cord passing round a smooth iron bar has its two parts inclined at 60° and is strained with a tension of 30 lbs; find the pressure on the bar. *Ans.* $30\sqrt{3}$ lbs.

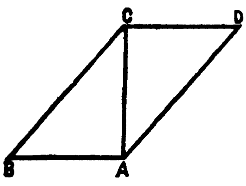
39. Shew that forces P and P + Q acting at an angle of 120° have their resultant the same in magnitude as Q and P + Q acting at the same angle. (The resultant in both these cases is $\sqrt{P^2 + PQ + Q^2}$.)

40. The side BC of an equilateral trangle ABC is bisected in D and forces are represented in direction and magnitude by BA, BD. Find the magnitude of their resultant, if the force along BD be equal to the weight of one pound. *Ans.* $\sqrt{7}$ lbs.

41. A weight of 10 lbs. is supported by two forces, one of which acts horizontally and the other at an angle of 30° with the horizon; find the magnitude of the forces. *Ans.* 20 lbs.; $10\sqrt{3}$ lbs.

42. If the resultant is at right angles to one of the forces, show that it is less than the other force.

Let AB be one of the two component forces, and let AC the resultant be at right angles to AB. Join BC and complete the parallelogram ABCD.



Then by the 'parallelogram of forces,' AD is the other component, which is equal to BC. (Eu. I., 34.)

Now, in the right-angled triangle ABC,

$BC^2 = AB^2 + AC^2$ (Eu. I., 47),

∴ BC^2 is greater than AC^2,

and BC greater than AC;

but AD = BC,

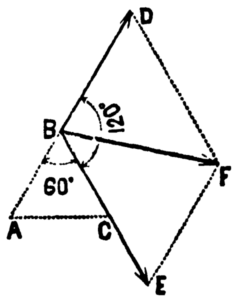
∴ AD is also greater than AC,

or AC is less than AD;

i.e., the resultant is less than the other force.

43. If two forces be inclined to each other at an angle of three halves of two right angles, find the ratio of their magnitudes when the resultant equals the less. *Ans.* 1 : $\sqrt{2}$.

44. Forces of 4 lbs. and 5 lbs. act along the sides AB and BC of an equilateral trangle, find the resultant.



As the forces act along the sides of an equilateral trangle, the angle at which they meet at B is 60°, but the first force of 4 lbs. acts from A to B, *i e., towards* the apex B of the triangle, while the second force of 5 lbs. acts from B to C, *i e., away* from it. They both meet at B and their directions and magnitudes will, therefore, be properly represented by the lines BD and BE, these being in the ratio of 4 : 5.

The angle EBD between them is of 120° and the problem is *to find the resultant of two unequal forces acting at an angle of* 120°

Applying, therefore, the formula

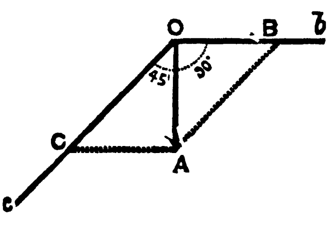
$$R^2 = P^2 + Q^2 - P.Q. \left(\text{Art. 23. (3) } (a) \right)$$

By substitution we get

$$R^2 = 16 + 25 - 20$$
$$= 21,$$

and $R = 4{\cdot}58$ lbs. nearly.

45. It is required to substitute, for a given vertical force P, two forces, one horizontal and the other inclined at an angle of 45° to the vertical; determine the magnitude of these two forces.



Let OA represent P in magnitude and direction.

From O draw Ob at right angles to OA and also draw Oc making an angle of 45° with OA. From A draw AB parallel to OC meeting Ob in B and AC parallel to OB meeting Oc in C.

Then OA is the diagonal of the parallelogram OBAC and by the 'parallelogram of forces,' OB and OC are the components of OA.

Now as angle AOC is 45°, therefore angle OAB is also 45°, and as angle AOB is 90°, therefore in triangle AOB the third angle ABO is 45° and OB = OA; hence the horizontal component is equal to P.

Again the component OC is equal in magnitude to AB,

$$
\begin{aligned}
\text{and } AB &= \sqrt{OA^2 + OB^2} \\
&= \sqrt{2\,P^2} \\
&= P\sqrt{2}.
\end{aligned}
$$

Therefore the horizontal component is P and the other component $P\sqrt{2}$.

46. Three equal forces act upon a point; the angle between the first and the second is 30° and that between the second and the third is 60°: find the resultant.

For an angle of 60° $R^2 = P^2 + Q^2 + P.Q$, therefore for the equal forces two and three $R^2 = P^2 + P^2 + P^2$,

$$
\begin{aligned}
&= 3\,P^2, \\
\text{and } R &= P\sqrt{3}.
\end{aligned}
$$

Now as the two forces are equal, the resultant is equally inclined towards both, *i. e.*, the angle between the second force and the resultant is of 30°; and as the angle between the first force and the second is also 30°, therefore the angle between the first force and the resultant of the second and the third is 60°. Therefore again applying the formula

$$
\begin{aligned}
R^2 &= P^2 + Q^2 + P.Q, \\
\text{we get } &= P^2 + 3\,P^2 + P.P\sqrt{3} \\
&= 4\,P^2 + P^2\sqrt{3} \\
&= P^2\,(4 + \sqrt{3}) \\
\text{and } R &= P\sqrt{4 + \sqrt{3}} \\
&= P\sqrt{4 + 1\cdot 73} = P\sqrt{5\cdot 73} \text{ approximately.}
\end{aligned}
$$

Note.—We first obtained the resultant of the second and the third forces, because if we had first compounded the first and the second forces, then their resultant would have made an angle of 15° with the second force and the angle between it and the remaining third force would have been one of 75°.

47. Which will be more advantageous to employ, two forces of 9 lbs. and 12 lbs. acting at right angles, or two forces of 15 lbs. acting at an angle of 120°?

When the two forces are at right angles to each other, then $R = \sqrt{P^2 + Q^2}$ (*vide* Article 23, 1);

by substitution, therefore,

$$R = \sqrt{9^2 + 12^2} = \sqrt{225} = 15.$$

When equal forces make an angle of 120° between them, then

$$R = P = Q = 15 \quad (\textit{vide}\ \text{Art. } 24).$$

Therefore, the effect in both cases is the same.

48. Two forces 12 lbs. and 20 lbs. act on a particle and produce a resultant 28 lbs.; at what angle are the forces acting ?

(*Vide* summary at the end of Article 23.)

$$R^2 = P^2 + Q + PQ\sqrt{x},$$

by substitution

$$(28)^2 = (12)^2 + (20)^2 + 12 \times 20\sqrt{x}$$

$$784 = 144 + 400 + 240\sqrt{x}$$

$$\therefore x = 1$$

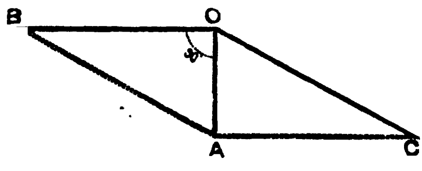
The quantity under the radical sign is one and the sign is positive, therefore, the angle between the forces must be 60°.

49. Find the angle between the two forces of 3 lbs. and 5 lbs. so that they may be kept in equilibrium by a force of 7 lbs. *Ans.* 60°.

50. Determine the angle between two forces of 42 lbs and 112 lbs. that they may be balanced by a force of 98 lbs. *Ans.* 120°.

51. If the resultant of two forces is at right angles to one force and equal to half the other, compare the forces.

By hypothesis OC is twice OA and OA is at right angles to OB.



$$\therefore AC^2 = OC^2 - (\tfrac{1}{2}\ OC)^2$$

$$= \tfrac{3}{4}\ OC^2,$$

$$\text{and } AC = \frac{\sqrt{3}}{2}\ OC,$$

$$\text{but } OB = AC,$$

$$\therefore OB = \frac{\sqrt{3}}{2} OC,$$

$$\text{hence } OB : OC :: \sqrt{3} : 2.$$

52. The magnitudes of two forces are as 3 : 5, and the direction of their resultant is at right angles to that of the smaller force; compare the magnitudes of the larger force and the resultant. *Ans.* 5 : 4.

53. The magnitudes of two forces are as 1 to 2; at what angle must they act upon a particle, that their resultant may be at right angle to one of the forces. *Ans.* 120°.

54. The sum of two forces is 18, and the resultant, which is at right angles to the lesser of the two, is 12; find the magnitude of the forces. Ans. 13, 5.

55. If one of two forces acting on a particle is 5 kilogrammes, and the resultant is also 5 kilogrammes and at right angles to the known force, find the magnitude and direction of the other force. *Ans.* 7·07 kilog. 135°.

56. The difference between two forces is 17, and the resultant which is at right angles to one of them is 25, find the forces. *Ans.* $9\frac{15}{17}$, $26\frac{15}{17}$.

57. If the resultant of two forces is at right angles to one force, and equal to half the other, compare the forces. *Ans.* $2 : \sqrt{3}$.

58. If the resultant is at right-angles to one of the forces and equal to it, show that the other force makes an angle of 135° with it.

59. If the resultant of two forces acting at an angle of 135° be 25 and at right angles to one of the forces; determine the forces. *Ans.* 25, $25\sqrt{2}$.

60. The resultant of two forces acting at an angle of 120° is at right angles to one of them; this force is 20; determine the other force and the resultant. *Ans.* 40, $20\sqrt{3}$.

61. Find the resultant of two forces acting at an angle of 45°, one of which is twice the other. *Ans.* $\sqrt{(5 + 2\sqrt{2})}$ times the smaller force.

62. If two equal forces P and P, acting at an angle of 60°, have the same resultant as two equal forces Q and Q, acting at right angles, show that P is to Q as $\sqrt{2}$ is to $\sqrt{3}$.

63. Three equal forces act in one plane on a point in such a way that each of them makes an angle of 120° with each of the other two; prove that the forces will balance.

64. Three forces, each of 10 lbs., act upon a point, the angle between the first and the second is 30°, and the angle between the second and the third is 60°; find the resultant. *Ans.* 23·9 lbs.

65. Three forces, 8, 8 and 6 lbs., act upon a point making angles of 120 with each other. What force must be added to the 6 lbs. to preserve equilibrium? *Ans.* 2 lbs.

66. Four equal forces act on a point; the first is at right angles to the second, the third is at right angles to the resultant of the first two and the fourth is at right angles to the resultant of the other three; find the resultant of all four. *Ans.* 2 P.

67. Three forces, 10, 10, 36, act on a point at angles of 120°: find the resultant. *Ans.* 26.

68. Three forces 20, 30 and 40 act on a point at angles of 120° between them; find their resultant.

If OA, OB and OC are the directions along which the forces act, then as three forces of 20 each are in equilibrium they can be removed by the principle of superposition of forces (article 12), and there will remain only two forces, one of 10 along OB and another of 20 along OC and the resultant of 10 and 20 at an angle of 120° is $10\sqrt{3}$. *Ans.* $10\sqrt{3}$.

CHAPTER III.

Parallel Forces.—Centre of Gravity

26. We shall now proceed to consider the conditions of equilibrium of *parallel* forces acting at different points of a body. Up to now we have dealt with the equilibrium of forces *acting at a single point*, *i. e.*, acting on a particle. The cases that we shall now discuss will be of bodies of much more extended size, and we shall suppose these bodies to be so rigid that no force can distort their form or change their shape. It is necessary to consider them thus absolutely *unyielding* in order that the distances between the various forces may remain unchanged.

Def.—A **rigid body** is one whose size and shape remain the same, whatever forces be applied to different parts of it.

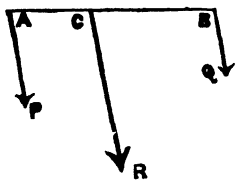
The above is merely a conception, absolutely rigid bodies do not exist in nature.

Def.—Parallel forces acting in the **same direction** are called **like parallel forces.**

Def.— Parallel forces acting in **opposite directions** are called **unlike parallel forces.**

27.—Resultant of two like parallel forces acting on a rigid body. Experimental proof.

When two like parallel forces act on a body, their combined **effect** is equal to a single force equal **in magnitude** to their **sum, parallel** to their **direction** and **applied at a point** which will **divide the line** joining the two forces **in the inverse ratio*** of the magnitude of those forces.



Suppose two forces P and Q to be acting on a body at A and B.

Then R will be the resultant such that $\mathbf{R = P + Q}$, parallel to either of them and situated at a point C such that it will divide the line AB into two parts CA and CB, so that

$$\mathbf{CA : CB :: Q : P} \, ;$$

$$\text{hence } P \times CA = Q \times CB.$$

*** Two quantities A and B are said to be in the *inverse ratio* of C and D, when A : B :: D : C.**

When four quantities are in proportion, the product of the extremes is equal to the product of the means. Thus in the preceding proportion $A \times C = B \times D$.

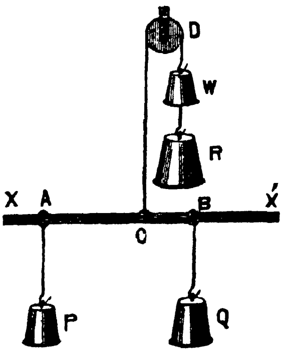
Experimental Proof.—We have to prove that the resultant

(i) is equal to $P+Q$,

(ii) parallel to P and Q,

and (iii) its points of application divides the distance between P and Q in a definite ratio.

(*a*) *Mechanical details.*—Take a straight uniform rod XX, suspend it at its centre C by a thin cord that passes over a pulley D and hang a weight W, equal to the weight of the rod, from the other end of the cord; then the rod will remain horizontal and in equilibrium.



Next take two weights P and Q, hang them from any two points A and B on the rod and let a weight **R equal to their sum** be attached below W.

(*b*) *Observed facts.*—It will be then observed that the effect of R will be to lift up the rod and R will thus act opposite to P and Q.

Next shift the weights P and Q on the rod till it becomes horizontal, and measure their distances CA and CB from the centre. It will be found that in one position and one *only* of the weights P and Q, the rod is horizontal and that

$$CA : CB :: Q : P.$$

(*c*) *Deductions*—

(i) R is evidently the anti-resultant of P and Q; therefore, the *magnitude of the resultant* is also equal to R which is the *sum of P and Q.*

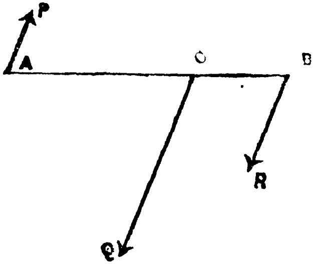
(ii) The anti-resultant acts along the vertical line CD, therefore the resultant also acts in the vertical direction and hence *parallel to P and Q.*

(iii) C, the point of application of the resultant, divides the distance AB in the definite ratio $\frac{CA}{CB} = \frac{Q}{P}$.

28. Resultant of two unlike parallel forces acting on a rigid body. Experimental proof.

When two unlike parallel forces act on a body, their combined **effect** is equal to a single force equal **in magnitude** to their **difference, parallel** to their **direction,** acting **towards** the side on which **the greater force** acts and **applied at a point** on the line joining the two forces produced, **such that the whole** produced line shall be **to the produced part inversely as the forces.**

Suppose the two unlike forces P and Q to be acting on a



body at A and C, respectively. Then (if Q be greater than P) **R = Q − P**, parallel to either of them and situated at a point B on the line AC produced such that

$$AB : CB :: Q : P,$$

and hence $P \times AB = Q \times CB$.

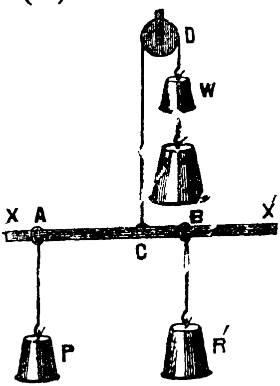
Experimental Proof.—We have here to prove that the resultant

(i) is equal to Q – P,

(ii) parallel to both P and Q,

and that (iii) its point of application is definite.

(*a*) *Mechanical details.*—As before, let the weight P



hang from A, and let the weight Q greater than P be fastened to the end of the string that passes over the pulley ; then P and Q are two unlike forces, of which Q is greater than P.

(*b*) *Observed facts.*—It will be then found necessary to hang a weight *R′* from same point B to maintain equilibrium such that R′ = Q —P, and the distances AB and CB, when measured will be found to be such that AB : CB : : Q : P ; *in no other position will R′ keep P and Q in equilibrium.*

(*c*) *Deductions*—

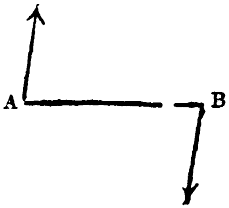
(i) R′ is the anti-resultant of P and Q ; therefore the *magnitude of the resultant* is also equal to R′, which is the *difference of P and Q.*

(ii) The resultant which is in the same line as the anti-resultant acts vertically and therefore parallel to both P and Q.

(iii) C, the point of application of the resultant, divides the distance AB in the definite ratio—

$$\frac{AB}{CB}=\frac{Q}{P}.$$

29. Couples.—When two unlike parallel forces are *equal* in magnitude, their resultant will be zero, and the effect of two such forces will evidently be to twist the body round in the plane in which they act. Two such forces from a *couple.*



Def.—Two equal and unlike parallel forces acting in different lines constitute a **couple.**

Def.—The line drawn at right angles to the directions of the two forces is called the **arm** of the couple.

30. If three forces are in equilibrium, then their lines of action must either be parallel or must meet in a point.

Let P, Q, and R, be the three forces in equilibrium.

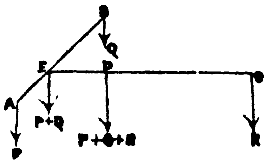
If two of them P and Q be parallel, they will have a resultant S, which must be parallel to them (articles 27 and 28), and as the body is in equilibrium this resultant must also be equal and opposite to R. Therefore the direction of R is parallel to the directions of P and Q ; in other words, all the three forces are parallel.

Next suppose that the lines of action of P and Q meet in a point O ; then their resultant S will pass through this point. (*Vide* Parallelogram of Forces.) But as the body

is in equilibrium this resultant S must also be equal and opposite to R, *i. e.*, the line of action of R must meet the line of action of S in O and therefore also of P and Q in O. In other words the lines of action of all the three original forces must meet in a point.

31. Resultant of any number of parallel forces. Centre of parallel forces.

Let P, Q, R . . . be any number of parallel forces, acting at points A, B, C . . . in a rigid body.



Join AB and on it take a point E, such that P : Q : : BE : AE ; then (by article 26) the forces P and Q, are equivalent to a single resultant force P + Q at E, acting parallel to both of them.

Next compound this resultant with the force R at C. Join EC and on it take a point D, such that

$$P + Q : R : : CD : ED,$$

then, as before, the forces $P+Q$ and R are equivalent to a single resultant force $P+Q+R$ at D, acting parallel to them all. Any number of forces can thus be reduced to a single force.

The point of application of the resultant force is called the **centre** of the parallel forces ; it is determined only by the magnitudes and points of application of the component forces. In the figure, E is the 'centre' of P and Q and D is the 'centre' of P, Q and R, and the position of E is determined by the magnitudes of P and Q and their distance apart and that of 'centre' D by the

magnitudes of P+Q and of R and the distance between them.

If the forces be in one straight line, then the centre of the forces will be also in the same line. Because the resultant of P and Q will be in the line joining P and Q, *i. e.*, in the original straight line and the resultant of P + Q and of R will be in the line joining them, *i. e.* again in the original straight line and so on.

32. The moment of a force about a point.

The greater the force, the greater is the tendency of a body to turn round a given point. Moreover the greater the distance* at which the force is applied from the point, the greater also is this tendency ; for example, it is easier to just open a door by pressing it somewhere near the handle than near the hinge. Also in articles 27 and 28 it has been pointed out that the smaller forces act at greater distances from the point C in order to preserve equilibrium.

The tendency to turn, therefore, depends on—

(*a*) the force applied and (*b*) the distance of its line of action from the fixed point.

A force of 2 lbs. at the distance of 2 ft. will have the same effect as a force of 4 lbs., acting at the distance of 1 ft; hence *the tendency* to turn is *to be measured by the product of the force and distance. This power to turn a body is called the moment of the force.*

Def—The **product of a force** into the **perpendicular** distance of its *direction* from any given point is the **moment** of the force about that point.

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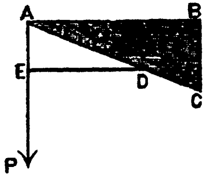
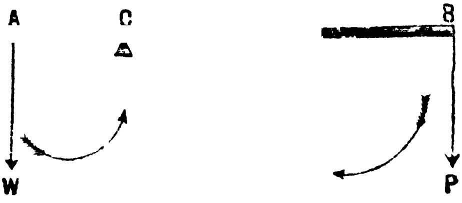
* **By "*distance*" is meant the *shortest distance, i. e.*, the *perpendicular distance.***

For instance, let the force P act on the body ABC at A, then the moment of P about B is P × BA, where BA is the perpendicular dropped from B on the direction of P; and the moment of P about any other point D is P × DE, where DE is the perpendicular dropped from D on the direction of P.



Note, therefore, that the moment of a force never vanishes except when the point about which the moment is taken is on the line of action of the force. In the above figure, the moment will only vanish if taken about A, as no perpendicular can be drawn from A on the direction or line of action of P, or in other words as the perpendicular from A on the direction of P is zero, the moment must have zero value.

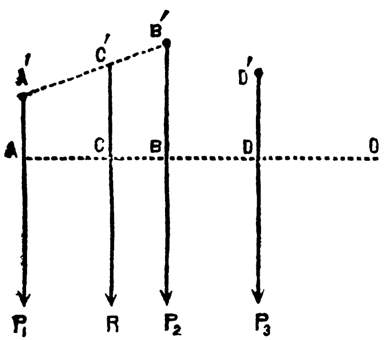
In the above figure, if the block of wood ABC were pivoted at D, then the force P could turn it from the left to the bottom of the book, thence to the right, next towards the top of the book, and finally back to its original position and so on, or in a direction contrary to the hands of a watch. The force which has a **tendency to turn** a body **contrary to** the direction of the **hands of a watch** is said to have **positive moment**; if the tendency be in the opposite direction, *i. e.*, **to turn** the body **in the direction of the hands** of a watch, than the **moment** is said to be **negative**.



In the figure, the rod AB is capable of turning about C in two directions. The tendency to move under the action of P is said to be negative as it is in the direction of the hands of a watch and the moment of P round C has a negative value; while the tendency to opposite motion under W is said to be positive and the moment of W round C has a positive value.

33. Moment of the resultant of parallel forces.

Let parallel forces P_1, P_2, P_3 . . . act at points A', B', D' . . . respectively in a rigid body. From some point O drop the common perpendicular ODBA on the directions of these forces.



Now $P_1 \times C'A' = P \times C'B'$ and R the resultant of P_1 and P_2 acts at C'. (*Vide* Article 27.)

By the principle of 'transmissibility of forces,' next transfer these forces to points A, B and C respectively on the lines of action of P_1, P_2 and R. (*Vide* Article 13.)

$$\text{Then, } P_1 \times CA = P_2 \times CB,$$
$$\text{and } P_1 \times CA - P_2 \times CB = 0:$$
$$\text{add to each side } R \times OC,$$
$$\text{then } P_1 \times CA - P_2 \times CB + R \times OC = R \times OC,$$
$$\text{but } R = P_1 + P_2,$$
$$\therefore P_1 \times CA - P_2 \times CB + P_1 \times OC + P_2 \times OC = R \times OC;$$
$$\text{hence } P_1(CA + OC) + P_2(OC - CB) = R \times OC,$$
$$\text{and } P_1 \times OA + P_2 \times OB = R \times OC.$$

Therefore, the moment of the resultant R round any point O is equal to the sum of the moments of the components P and Q round the same point.

Similarly by combining R with P_3 and that resultant with a new force, we can prove that if P_1, P_2, P_3 . . . be any number of parallel forces and T their resultant, and P_1x_1, P_2x_2, P_3x_3 and Tx their respective moments round any point O, then

$$Tx = P_1x_1 + P_2x_2 + P_3x_3 + \quad . \quad . \quad .$$

Hence, the **moment of the resultant** of a number of parallel forces about any point is **equal to the sum of the moments of its components.**

34. Condition of equilibrium of parallel forces.

We will now proceed to show under what conditions a set of parallel forces is in equilibrium.

If P_1, P_2, P_3 . . . be parallel forces, then their resultant T is equal to their sum ; and to maintain equilibrium a force S equal to T but acting in the opposite direction must be applied at the point of application of T and the moment of S will be equal in numerical value to the moment of T but of opposite sign ; and the condition of equilibrium of S, P_1, P_2, P_3 . . . will be

$$Sx - P_1x_1 - P_2x_2 - P_3x_3 \quad . \quad . \quad . = 0.$$

Therefore, parallel forces are in **equilibrium when the algebraical sum of their moments** round any point equals **zero.**

35. Centre of gravity.

We have seen in article 6, that the force with which gravity acts on a particle is called its weight. Now a body

is composed of a number of particles, therefore the weight of the body is the total or resultant-force of gravity acting on the body.

We moreover know that the directions in which the force of gravity acts on neighbouring particles are parallel to one another (*vide* article 8), therefore the resultant or total weight is equal to the sum of the weights of all the particles composing the body (article 31) and the problem of determining the centre of gravity of a body is the same as that of determining the centre of parallel forces. Just as the resultant of parallel forces passes through the 'centre of the forces' (*vide* definition, article 31), so does the resultant force of gravity *i.e.*, the whole weight of a body is supposed to act at the 'centre of gravity.'

Def.—The **centre of gravity** of a body is that point through which the resultant or total force due to the earth's attraction passes.

36. A body can have only one centre of gravity.

For if possible, let it have two such centres G and G′ and let the body be turned till the line joining G and G′ is horizontal. Then the resultant of the parallel vertical forces will pass vertically through G and also vertically through another point G, which is in the same *horizontal* plane as G. This is absurd, because the resultant cannot act in two different directions at the same time.

37. The point of suspension and the centre of gravity of a body both lie in the same vertical line.

By the definition of C. G. it is evident that the resultant of the weights of all the particles of a body always passes

through the C.G. in whatever position the body may be placed, and this resultant-weight must act in the direction of the plumb line ; moreover the resultant-weight is supported at the point of suspension ; therefore, it is clear that the line joining the point of suspension with the C.G. must be vertical.

38. Experimental determination of centre of gravity.

Bearing in mind the two important principles established in articles 36 and 37, we can easily determine the C.G. of a body. At first we will deal with *thin uniform* rods, rings and laminæ, their geometric analogues being straight lines, circles and plane surfaces.

(*a*) *A thin uniform rod.* Suspend it from one end, then it will evidently hang vertical and the C.G. will lie in its length. Next suspend it by a long loose string, the two ends of which are tied to the two extremities of the rod and drop a vertical line* from the point of suspension ; then the C.G. is in this line (article 37). Evidently, therefore, the C.G. is in the intersection of this line with the length of the rod. It need hardly be here pointed out that the C.G. will be in the middle of the rod.

(*b*) *A ring of thin uniform wire.* Suspend it from any point in its circumference, the ring will hang with its plane vertical and the prolongation of the direction of the string, (which is the direction of the plumb line), will divide the ring into two equal halves ; the C.G. will lie in this line. Again suspend it from some other point in its circumference, then as before the C.G. will lie in the new vertical line. The C.G. of the ring, therefore, is at the intersection of the two lines, *i.e.*, at the geometrical centre of the ring.

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* This may be done by noting the direction of a plumb line suspended from the same point as the rod.

(*c*) *A thin uniform circular lamina.* Proceed as with (*b*) and the result will be the same.

(*d*) *A thin uniform lamina of any shape.* Hang it from any point and draw a vertical line from the point of suspension by the help of a plumb line or otherwise; the C.G. will lie in that line (art. 37). Next hang the body from some other point and proceed as before; then the C.G. will lie in that line. Now as a body can have only one C.G. (art. 36.), therefore, it evidently lies in the point of bisection of the two vertical lines.

Generally, therefore, the *mode of procedure* is the same for all bodies of whatever shape, thick or thin, uniform, or not uniform, but it must be remembered that unless the body be *very thin*, the vertical lines will, as a rule, pass inside the bodies and the C.G. will lie not on the surface but within it.*

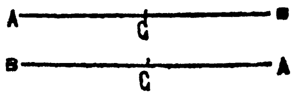
39. Determination of the centre of gravity by calculation.

We shall consider the following more important cases:—

(*a*)	The C.G. of a thin uniform rod.			
(*b*)	,,	,,	,,	ring.
(*c*)	,,	,,	,,	disc.
(*d*)	,,	,,	,,	parallelogram.
(*e*)	,,	,,	,,	homogeneous sphere.
(*f*)	,,	,,	,,	right cylinder.
(*g*)	,,	,,	,,	parallelopiped.

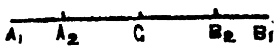
* For instance determine the C.G. of a ruler and of a billiard cue and compare results with (*a*).

(*a*) **The centre of gravity of a thin uniform rod is its centre.**—*Proof.* Let AB be the rod and let its centre of gravity be at G; turn round the rod, so that *B* now



occupies the position formerly occupied by A, then A will occupy the position of B; the particles in AB are in no way disturbed, therefore the position of the centre of gravity relatively to A and B, will remain as before, this will only happen if it be mid-way between A and B.

Alternative proof. The rod being thin, it may be supposed to be made up of a single series of particles and the particles are all of equal weight because the rod is uniform and homogeneous. Take the two equal extreme particles, A_1



and B_1, then the resultant of their weights will pass through the middle point G (*vide* article 27); similarly the resultant of the next two particles A_2 and B_2 will also pass through the same point and thus we can exhaust all the particles of which the rod is composed; in every instance the resultant of the pair of particles will pass through the middle point G and the point where the resultant weight acts or the C. G. of the rod will be in its middle point G.

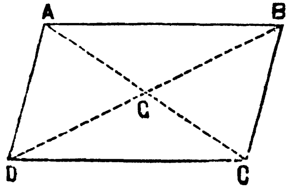
(*b*) **The centre of gravity of a uniform ring is its centre.**—*Proof.* Rotate the ring on its centre of figure, then as the ring will continue to occupy the same space, its C.G. must remain unchanged. But there is only one point within the ring which remains fixed in position when the ring rotates and that is its centre of figure; therefore, the centre of figure is the C.G. of a uniform ring.

Alternative proof. The ring may be supposed to be formed by a row of equal number of particles. Draw any diameter then, by article 27, the resultant of the weights of the particles at the opposite ends of the diameter will be at the centre and this will be true of every diameter and therefore the resultant weight of the whole ring will be at its centre; in other words, the C.G. will be at the centre of the ring.

(*c*) **The centre of gravity of a thin uniform circular disc is at its centre.**—*Proof.* The same as for (*b*).

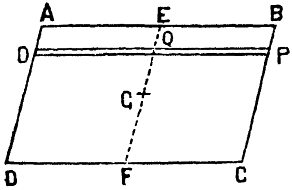
Alternative proof. Divide the disc into large number of thin concentric rings; then the C.G. of each ring will be at the centre of the disc, and, therefore, the C.G. of the whole disc will be also at the centre.

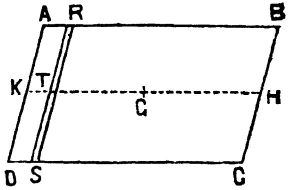
(*d*) **The centre of gravity of a thin uniform parallelogram is at the point of intersection of its diagonals.**—*Proof.* Let the parallelogram be ABCD, join AC, then the diagonal AC will divide the parallelogram into two equal parts, therefore the C. G. will be in the diagonal AC.



Next join BD, then this diagonal will again divide the parallelogram into two equal parts and therefore the C. G. of the parallelogram will be in BD. The C. G. therefore is in the point of intersection of AC and BD.

Alternative proof. Divide the parallelogram ABCD into a number of very thin strips like OP all parallel to AB ; then the centre of gravity of OP is at its middle point Q and similarly the centre of gravity of all other strips will be at their middle points, and the centre of gravity of the whole parallelogram will be on the line EF midway between AD and BC.

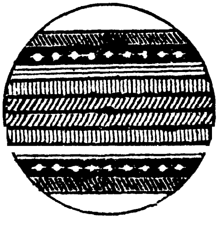




Again, if the parallelogram be divided into strips parallel to AD, it can be similarly shown that its C. G. will be in the line KH midway between AB and DC. The C. G. of the parallelogram, therefore, is at G, the point of intersection of EF and KH. Note that this is also the intersection of the diagonals.

(*e*) **The centre of gravity of a homogeneous sphere is its centre.**—*Proof.* Similar to the first proof for a ring.

Alternative proof. A sphere may be supposed to be formed of pairs of thin discs gradually getting smaller and smaller in diameter. The C. G. of each disc will be at its centre and of pairs of similar equi-distant discs at the centre of the sphere ; therefore, the C. G. of all the discs, *i.e.* of the whole sphere will be at its centre.



(*f*) **The centre of gravity of a homogeneous right cylinder is at the centre of its axis.**

Proof. Rotate the cylinder, so that it continues to occupy the same space, its C. G. will then remain unchanged ; but there is only one line in the cylinder which remains fixed and that is the axis of the figure, therefore the C. G. lies in that axis. Now if the cylinder is turned end for end, the particles of which the cylinder is composed are in no way disturbed, therefore the position of the C.G. in the axis relatively to the two ends of the cylinder must remain the same as before and this will only happen if the C.G. be midway between the two ends of the axis.

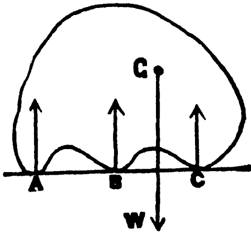
(*g*) **The centre of gravity of a homogeneous rectangular parallelopiped is in the middle of the line joining the points of intersection of the diagonals of two opposite faces.**—*Proof.* We may suppose the parallelopiped to be formed of a number of thin uniform parallelograms. Now the C.G. of each parallelogram, will be at the intersection of its diagonals and taking pairs of parallelograms equally distant from the two opposite surfaces, it is evident that the C.G. of the pair of parallelograms will be in the centre of the line joining the centres of gravity of the two parallelograms and this will be true of all other pairs of parallelograms. Therefore the C.G. of the whole set of parallelograms, *i.e.*, of the parallelopiped, will be in the centre of the line joining the points of intersection of the diagonals of opposite faces.

40. Equilibrium of a body resting on its base.

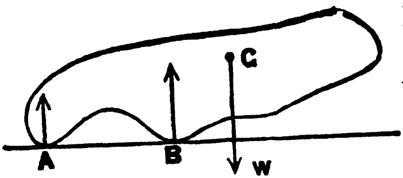
Def. When a body rests on a plane, its **base** is the area enclosed on the plane by a string drawn tightly round all the points in which the body touches the plane.

A body resting on a plane surface is in equilibrium under two forces, first its weight acting through its centre of gravity and second the re-action of the surface or the upward pressures at the points of contact ; this resultant re-action must act in the same line as the weight, otherwise the weight, *i.e.*, the downward pull due to gravity, will not be counter- balanced and the body will fall.

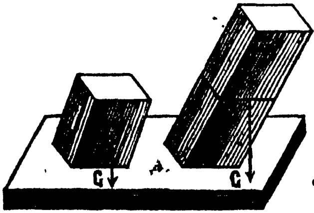
Let the figure be a vertical section of a stone resting on the ground at points A, B and C ; the resultant of the re-actions at A, B and C will evidently lie in the line A C (art. 27), that is, within the base and, therefore, to maintain equilibrium the C. G. of the body must also be in the same line.



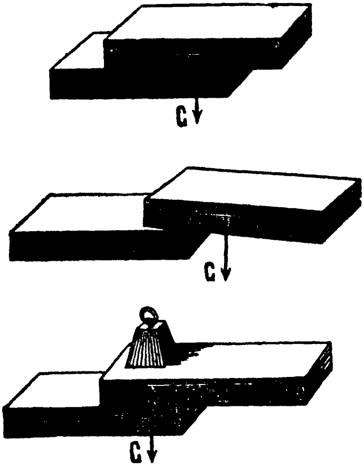
If, however, the C. G. lies outside the base as in the second figure, then there can be no equilibrium.



This may be demonstrated experimentally by taking two proper parallelopipeds, any one of which can stand firmly on the table as the C.G. falls within the base ; if the parallelopipeds, however, be securely fixed one on the other, they will no longer stand, because the C.G. of the compound parallelopiped is higher than that of either singly and the vertical falls outside the base.



Or again, a flat piece of wood or a brick may be placed on the top of another and provided that the C.G. of the top block falls within the area of the lower supporting block it will remain in position; but if it falls outside it the upper piece will fall and to prevent it from falling it will be necessary to load it, as in the third figure, in order to bring the centre of the mass inside the base.



The rule, therefore, is that *a body will stand or fall according as the vertical line through its C. G. falls within or without the base on which it rests.*

Numerous examples of this principle, that it is essential for the stability of a body that the vertical should fall within its base, daily present themselves to our notice. Thus a man carrying a load on his back leans forward, while a seller of sweetmeats who carries a tray in front of him leans backward, the reason in each case being that the fact of inclining the body brings the line of direction within the base formed by the feet. When a man stands on one leg, he at the same time inclines his body so as to bring much of it over the leg on which he rests, and therefore, in walking the body is alternately inclined to the side which is supported by the foot. It will be impossible for a man to stand on one leg if he does not incline his body; thus if a man stands close to a vertical

wall and rests his head and one leg against the wall, he will find that he cannot stand on that leg, because he is unable from his position to bring his centre of gravity vertically over it. A cart loaded high with hay has its C. G. high up and if one wheel gets tilted by accidently passing over a big stone, it is very liable to be overturned as the vertical through the C. G. may fall outside the wheels. Another cart as heavily loaded with stone or iron will not so easily turn over because the C. G, being low within the cart, the vertical with the same amount of tilting will not be so much displaced from the central position. The most remarkable case of this kind in which a structure stands uninjured though tiltled away from the vertical position is that of the Leaning Tower at Pisa. A plumb-line dropped from the highest point of the tower falls several feet outside the base, yet the tower stands firm because *the vertical through its C. G. is well within the base.*

41. States of equilibrium;—stable, unstable and neutral.—For equilibrium it is only necessary that the vertical through the C. G. should pass through the base, hence if the body is supported at one point only then the vertical should pass through this point. The C. G. may therefore be either vertically *below*, or vertically *above*, or *at* the point of support; in all these cases the body will be in equilibrium, but there will be great difference in the permanency or stability of the equilibrium.

Let us take an instance of the first kind, such as a weight hanging by a cord. Now remembering that the tendency of the force of gravity is to bring the C. G. of a body as near to the surface of the earth as possible, it will be evident that if we displace the weight it will swing to and fro and finally assume the vertical position, which will be the nearest to the earth; and this may be repeated over and over again. This state of equilibrium is therefore called *stable.*

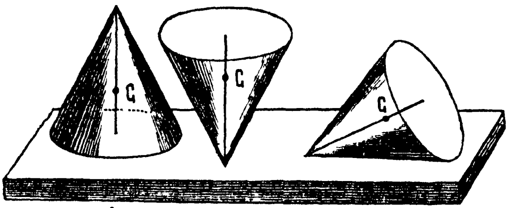
If we, however, take a stick and dexterously support it upright on the tip of the finger, then it will, of course, be in equilibrium, but the slightest shake or the least puff of wind will cause it to fall. This happens because the C. G. which is somewhere in the middle of the stick is vertically above the point of support, and the slightest tilting throws the vertical through the C.G. outside the supporting point, and this destroys the equilibrium. Such a state of equilibrium is, therefore, called *unstable.*

If we again take a uniform rod, drill a hole in the middle of its length, so as to psss it through the C. G. and put a stiff wire through the hole, then the rod will remain in any position; turn it as we like on the wire, the distance of the C.G. from the surface of the earth will remain constant and therefore there will be no tendency to occupy one position in preference to another. Similarly if we take a sphere, then it will be in equilibrium in *any* position, because its C. G. will throughout remain at the same distance from the surface on which the sphere rests. This kind of equilibrium is called *neutral.*

Def. A body is said to be in **stable** equilibrium when it tends to return to its original position of rest after being slightly disturbed.

Def. A body is said to be in **unstable** equilibrium when a slight disturbance tends to move it further and further away from its original position of rest or equilibrium.

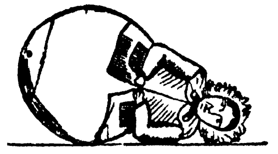
Def. A body is said to be in **neutral** equilibrium when on being displaced, it remains at rest in the new position.



A B C

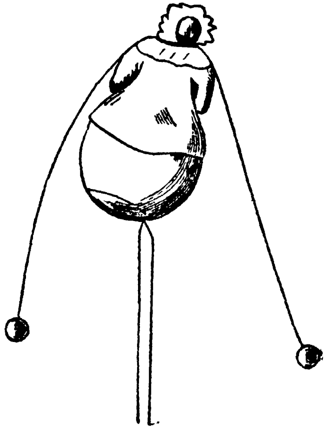
A cone in the three positions shown in the figure well illustrates the three states of equilibrium. In position A, the cone is represented resting on its base; if the cone be now slightly inclined from the vertical it offers a considerable resistance to displacement and, when permitted, immediately, returns to its first position; this is the position of *stable* equilibrium. Although theoretically it is possible to put the cone in position B, it is practically almost impossible to do so, because, although the vertical from the C. G. may be brought within the base on which the cone rests, which is in this case the vertex of the cone, yet the very slightest force, the action of the air for example, will be sufficient to displace the cone, and the least displacement will cause the vertical line to pass outside the point of support. The cone in this position is in a state of *unstable* equilibrium.

There is yet another position in which the cone could be made to rest, which is represented at C; as the centre of gravity of the cone is in the axis of the cone (*i. e.*, the right line joining the vertex with the centre of the base), it is clear that if the cone be rolled along the table on its side, the distance of the centre of gravity from the table is throughout constant, and the cone will rest equally well in any position; here the cone is in *neutral* equilibrium.



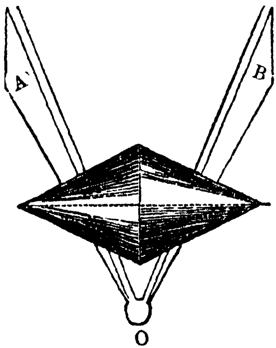
The toys called '*tumblers*', shown in the above figures are constructed on this principle. Were the toy uniform in substance, it would lie in the position shown in the first figure, but a piece of lead fixed inside its base so lowers the centre of gravity of the whole, that this point is in its lowest position when the figure is standing upright, and when we lay the toy down on its side, we raise the centre of gravity, which, on releasing the toy again occupies the lowest position it can possibly occupy.

The toy might even be balanced on a point if we could contrive to bring the centre of gravity below that point. This may be accomplished by hanging a wire, with a bullet on each extremity, round the neck of the figure. In this position the centre of gravity of the whole is so lowered as to be actually below the point of suspension, and every attempt to overturn the figures raises the centre of gravity, which always returns again to the lowest position.



An apparently paradoxical experiment in which a double cone, formed by the union of two cones base to

base, runs up an inclined plane depends on the same principle. The inclined plane is constructed of two pieces of wood A O and B O which meet at an angle at the point of junction O, this is the lowest point of the plane. Now if the cone be placed at the angle it will run up the plane ; this



can easily be accounted for if we consider the position of the centre of gravity of the double cone in its first position and when it has completed its journey up the plane. The centre of gravity of the double cone is the centre of the common base of the cones which compose it, and owing to the slope of the sides of these cones being greater than that of the plane, the centre of gravity is in reality lower when the cones are at the summit of the plane than when they are at its foot.

QUESTIONS.

1. What are *like* and *unlike* parallel forces ?

2. What is a *rigid* body ? Are there any really rigid bodies in existence ? If not, of what use is this conception in Statics ?

3. State the law of composition of two like parallel forces and give an experimental proof. Also enunciate the law for two unlike forces and describe an experiment in support.

4. What is a *couple* ? How does a couple act and why ?

5. Name the two possible conditions under either of which three forces acting in a plane can remain in equilibrium. Prove them.

6. Define the *moment* of a force. How is it measured ? When is the moment *positive* and when *negative*

7. How will you determine the resultant of several parallel forces?

8. What is 'centre of parallel forces' and what is 'centre of gravity'? Distinguish between the two.

9. Show that a body can have only one centre of gravity.

10. How will you demonstrate that the centre of gravity lies in the vertical line drawn from the point of suspension?

11. What do you understand by the term *thin* as applied to a rod or ring in Statics?

12. Give an experimental method of finding the centre of gravity of a body and explain the principles on which it is based.

13. How will you proceed to find practically the centre of gravity of a uniform thin ring, and a similar round disc? Where is the centre of gravity situated in each of these cases?

14. How will you obtain the same results theoretically? Also determine the centre of gravity of a thin uniform parallelogram, a sphere, a right cylinder and a parallelopiped.

15. Determine the condition of equilibrium of a body resting on its base.

16. Name the three states of equilibrium and distinguish between them.

EXAMPLES ON LIKE PARALLEL FORCES.

1. Two like parallel forces of 4 lbs. and 6 lbs. respectively act at the ends of a bar 5 feet long. Find the magnitude and position of the resultant.

(A) The magnitude of the resultant equals the sum of the forces.

Therefore it is $(4+6)=10$ lbs.

(B) To find its point of application. Let AB represent the bar and R the point at which the resultant acts.

Let $BR = x$ ft.; then $AR = (5-x)$ ft.

If 4 lbs. acts at A, 6 lbs. at B and the resultant at R, we must have

$$4 : 6 :: BR : AR$$
$$:: x : 5-x$$
$$\therefore 6\times x = 4\times(5-x)$$
$$i.e., \quad 6x = 20-4x$$
$$\therefore x = 2 \text{ ft.}$$

Ans. The resultant is parallel to the given forces, and acts at a point 2 ft. from the end at which the force of 6 lbs. acts.

2. If $P=4$ lbs. and $Q=5$ lbs. be two like parallel forces, and the distance between their points of application be 12 inches, find the position of the resultant. *Ans.* $6\frac{2}{3}$ in. from P.

3. Find the resultant of two like parallel forces of 5 lbs. and 7 lbs. acting at points 1 ft. apart. *Ans.* 12 lbs. acting 7 in. from the smaller force.

4. Find a force which will balance two like parrallel forces of 12 lbs. and 15 lbs. acting along lines distant 6 ft. 9 in. from each other. *Ans.* 27 lbs. acting 3 ft. from the force of 15 lbs.

5. Find where a force must be applied to a rigid bar to keep it in equilibrium when acted on by two like parallel forces of 7 lbs. and 5 lbs. acting at points on the bar 5 ft. apart. *Ans.* $2\frac{11}{12}$ ft. from the smaller force.

6. The larger of two like parallel forces is 9 lbs.; the resultant is 12 lbs. acting at a distance of 3 ft. from the smaller component; find the distance between the components. *Ans.* 4 ft.

7. Two coolies, a man and a woman, are to carry a box, weighing 150 lbs., suspended from a pole whose ends rest on their shoulders. How will you arrange the matter so that the woman may sustain only one-third of the weight?

$$\left.\begin{array}{c}\text{Pressure supported}\\\text{by the man}\end{array}\right\} : \left.\begin{array}{c}\text{Pressure supported}\\\text{by the woman}\end{array}\right\} :: 1 : 2,$$

therefore, by the law of "like parallel forces", the length of the pole must be divided by the load in the inverse ratio of the pressures. (*Vide* Article 27). Hence the distance of the load from the woman's shoulders must be twice as great as from the man's.

8. Two men carry a block of iron weighing 176 lbs. suspended from a pole 14 feet long; each man is 1 foot 6 inches from his end of the pole Where must the block hang in order that one man may bear $\frac{4}{5}$ of the weight borne by the other? *Ans.* $4\frac{8}{9}$ ft. from the man bearing the greater weight.

9. A man supports two weights slung on the ends of a stick 150 centimetres long placed across his shoulder; if one weight be two-thirds of the other, find the point of support, the weight of the stick being disregarded. *Ans.* 60 c. m. from the larger weight.

10. A rigid rod 7 ft. long supposed to be without weight rests on a fixed point 2ft. 6 in. from one end. To this end a weight of 18 lbs. is attached. What weight must be hung from the other end so that the rod may be horizontal. *Ans.* 10 lbs.

11. A person carries a box weighing 25 lbs., at the end of a stick over his shoulder. If the stick be $3\frac{1}{2}$ ft. long, and if his hand is 2 ft. from his shoulder, find the pressure on the shoulder. *Ans.* 40 lbs.

12. Resolve a force of 280 lbs. into two like parallel forces, acting at distances from it of 4 and 3 inches.

(Vide figure of article 27)

By the principle of like parallel forces,

$$P \times CA = Q \times CB$$

$$\therefore \quad P \times 4 = Q \times 3$$

$$\therefore \quad P = \tfrac{3}{4}\,Q.$$

$$\text{Again} \quad R = P + Q$$

$$\therefore \quad 280 = \tfrac{3}{4}\,Q + Q = \tfrac{7}{4}\,Q$$

$$\therefore \quad Q = 160$$

$$\therefore \quad P = 280 - 160 = 120$$

Ans. 120 lbs.; 160 lbs.

13. Resolve a force 20 in two like parallel forces, one of them three times as far from the given force as the other. *Ans.* 15 and 5.

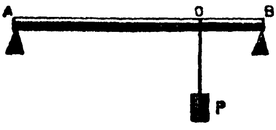
14. Two men carry a weight of 100 kilogrammes between them on a pole, resting on one shoulder of each ; the weight is three times as far from one as from the other : find how much weight each supports, the weight of the pole being disragarded. *Ans* 25 and 75 kilogrammes.

15. Two men, of the same height, carry on their shoulders a pole 2 metres long and a weight of 120 kilogrammes is slung on it half a metre from one of the men : what portion of the weight does each man support ? *Ans* 30 and 90 kilos.

16. A uniform beam 4 ft. long is supported in a horizontal position by two props which are 3 feet apart, so that the beam projects 1 ft. beyond one of the props ; show that the pressure on one prop is double the pressure on the other.

17. A beam, 40 feet long, is supported horizontally at its two extremities. Determine the additional pressure on each of the supports when a weight of 2,400 lbs. is hung at a point 10 feet distant from one end.

AB is the beam 40 feet long and the point O is 10 feet from the point of support B, and P is 2,400 lbs.



It is sufficiently plain that if the weight P were applied at the centre of the beam, each of the two supports would have to bear half the load. If the weight were placed exactly over one of the supports, then that support would have to bear the entire load, and the other support would have no pressure upon it beyond that due to the beam. In any other position of the weight, such as that shown in the figure, each support would have to sustain a certain share of the load.

The effect of each support is to press the beam upwards with a certain force; we can, therefore, determine the pressures on the supports, by the laws which regulate the equilibrium of a body acted upon by parallel forces (article 27).

For the beam to be in equilibrium, *the intensities of the two forces at the extremities of the beam must be in the inverse ratio of the segments into which the load divides the beam ;*

therefore, pressure at A : pressure at B :: OB : OA,

:: 10 : 30,

:: 1 : 3;

and as the sum of these pressures is 2,400 lbs., it is plain that the pressure at B due to the weight P must be 1,800 lbs., and that at A 600 lbs.

EXAMPLES ON UNLIKE PARALLEL FORCES.

1. Find the magnitude and position of the resultant of two unlike parallel forces of 4 lbs, and 5 lbs. acting at points 2 ft. apart.

(A) Magnitude of resultant.

$$R = P - Q$$
$$= 5 - 4 = 1 \text{ lb.}$$

The resultant is a force of 1 lb. acting in the direction of the larger force of 5 lbs.

(B) Point of application of resultant.

The rule found by experiment says that the resultant acts at a point in the line produced in the side nearer the greater force.

Let R be the point required in AB produced.

Let BR $= x$ ft.; then AR $= (2+x)$ ft. Then we have

$$4 : 5 :: BR : AR$$
$$:: x : (2+x)$$
$$\therefore \quad 5 \times x = 4(2+x)$$
$$i.e., \quad 5x = 8 + 4x$$
$$\therefore \quad x = 8 \text{ ft.}$$

Ans. 1 lb. acting at a distance of 8 ft. from the force of 5 lbs.

2. Two parallel forces of 3 and 4 units act on a body in opposite directions; specify the force required to balance them, and show by a diagram how the three forces act. *Ans.* 1 unit.

3. Parallel forces equal to weights of 9 lbs. and 4 lbs. act in opposite directions at points in a straight rod 10 inches apart; where must a force be applied to balance them? *Ans.* At a distance of 8 in. from the greater force.

4. Two parallel forces, acting upon a rigid bar, in opposite directions and at a distance of 4 in., are to each other as 5 to 7. Required the position of a third force, which will keep the two in equilibrium.

Ans. At a distance of 10 inches from the larger force and in the direction of the smaller force.

5. The resultant of two unlike parallel forces is 6 lbs., and acts at a distance of 8 in. from the greater force which is 10 lbs.; find the distance between them.

(Vide article 23.)

In unlike parallel forces,

$$R = Q - P$$
$$6 = 10 - P$$
$$\therefore \quad P = 4$$

Now, $P \times AB = Q \times CB$, where Q is the greater force of 10 lbs. and BC is 8 inches.

Let AC $= x$ in.; then AB $= (x+8)$ in.

$$P \times AB = Q \times CB$$
$$\therefore \quad 4 \times (x+8) = 10 \times 8$$
$$\therefore \quad 4x + 32 = 80$$
$$\therefore \quad x = 12 \text{ in}$$
$$\therefore \quad AC = 12 \text{ in.}$$

Ans. 12 inches.

6. The resultant of two unlike forces is 18 lbs. and acts 8 in. from the smaller force, which is 6 lbs.; find the distance between them. *Ans.* 6 in.

7. The resultant of two unlike parallel forces of 3 lbs. and 5 lbs. acts in a line at a distance of 5 ft. from the line of action of the lesser force, what is the distance between the lines of action of the two forces? *Ans.* 2 ft.

8. Parallel forces of 8 lbs. and 20 lbs. act in opposite directions at points in a straight rod 6 ft. apart. What must be the least length of the rod, that it may be balanced by a single force acting upon it? *Ans.* 10 ft.

9. If the resultant of two unlike parallel forces is 12 lbs., and acts at a distance of 5 inches from the greater and 7 inches from the lesser force, find the forces.

(Vide figure of Article 28)

By the principle of unlike parallel forces, if Q is greater than P,

then, $P \times AB = Q \times CB$

$\therefore \quad P \times 7 = Q \times 5$

$\therefore \quad P = \frac{5}{7} Q$

Again, $R = Q - P$

$\therefore \quad 12 = Q - \frac{5}{7} Q = \frac{2}{7} Q$

$\therefore \quad Q = 42$ lbs,

but $R = Q - P$

$\therefore \quad P = 30$ lbs. *Ans.* 42 lbs.; 30 lbs.

10. A force of 2 lbs. balances two unlike parallel forces acting at distances of 6 in. and 8 in. from it. Find the two forces. *Ans.* 6 lbs; 8 lbs.

11. Parallel forces are applied at two points 5 in. apart and are kept in equilibrium by a third force 3 in. from the one, and 2 in. from the other. What is the ratio of the forces? *Ans.* 2 : 3.

EXAMPLES ON LIKE AND UNLIKE PARALLEL FORCES.

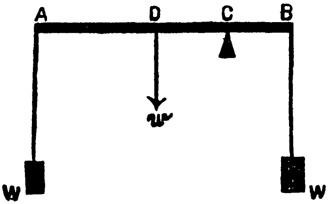
1. Two parallel forces acting at the extremities of a bar 19 ft. long have a resultant of 76 lbs. when acting in the same direction and one of 12 lbs. when acting in opposite direction. Find the forces and show where the resultants act. *Ans.* The forces are 44 lbs. and 32 lbs. The resultant ot 76 lbs. acts 8 ft. from the force of 44 lbs.; and that of 12 lbs. acts $50\frac{2}{3}$ ft. from the force of 44 lbs.

2. Forces of 5 and 7 units act in the same direction along parallel lines at points 2ft. apart. Where is their centre? If the direction of the former force is reversed where will now be their centre? *Ans.* 10 in. and 5ft. from the greater force.

3. If two like parallel forces of 20 lbs. and 30 lbs. be changed to 40 lbs. and 10 lbs. respectively, the lines of action being unaltered, shew that the distance, from the line of action of the 30 lbs. force of the new resultant is double that of the first resultant from the same line.

4. If two unlike parallel forces of 70 lbs. and 30 lbs. be altered to forces of 100 lbs. and 60 lbs., the lines of action being unaltered, shew that the distance of the new resultant from the force of 100 lbs. is double that of the original resultant from the same line.

5. A uniform beam 15 ft. long and weighing 8 lbs. has two weights, 12 and 18 lbs., suspended from its extremities and is balanced at a point on a prop. Determine the position of this point in the length of the beam and calculate the amount of pressure on the prop.



Let the 12 lb.-weight be suspended at A and the 18 lb.-weight at B, then by the "principle of parallel forces" (articles 27) the total down-ward pressure must be equal to $(12+8+18=)$ 38 lbs.; this is therefore the pressure on the prop.

Again, the weight of a body is supposed to act at its centre of gravity which in a uniform beam is at its centre (article 39); hence we may imagine the whole weight of the beam, *viz.*, 8 lbs., to act at its middle point D; we have, therefore, three weights, 12 lbs., 18 lbs., and 8 lbs., acting respectively at A, B and D, and by the "principle of the equality of moments" (article 33) the moment of the resultant must be equal to the sum of the moments of the three components round any point in the beam.

Let that point be A, then

$$38\times x=12\times 0+8\times 7\cdot 5+18\times 15,$$

where x is the distance of the prop, from the extremity A;

$$\therefore x=8\tfrac{13}{19}.$$

The position of the prop, therefore, is $8\frac{13}{19}$ feet from the extremity at which the 12 lbs. weight is suspended and the pressure on it is equal to 38 lbs.

6. Three weights of 2 lbs., 3 lbs., and 4 lbs., respectively, are suspended from the extremities and the middle point of a rod without weight; determine the point in the rod about which the three weights will balance. (See figure above.)

On the "principle of parallel forces," (article 27), the pressure on the fulcrum C is equal to the sum of the three forces $(2+3+4)=9$ lbs., and it acts opposite in direction to the three forces. Again for equilibrium the algebrical sum of the moments of all the forces round any point, such as C, must be zero (article 34); therefore taking moments round C—

$$2\times AC + 4\times DC + 9\times 0 - 3\times CB = 0$$
$$2\,AC+4\,(AC-AD)-3\,(AB-AC)=0$$
$$\text{but } AD=\tfrac{1}{2}\,AB,$$
$$\therefore 2\,AC+4\,(AC-\tfrac{1}{2}\,AB)-3\,(AB-AC)=0$$
$$9\,AC=5\,AB,$$
$$\therefore AC=\tfrac{5}{9}\,AB.$$

The point C is at five-ninths the distance of the length of the rod from the extremity A.

7. Forces of 5,—3, 4,—2, and 6 are arranged along a rod at equal distances (2 inches) :—find the resultant. *Ans.* R = 10 at a distances of $4\frac{2}{5}$ inches from 5.

8. A uniforn rod weighing 4 lbs. has 12 lbs. at one end, and 18 at the other. The centre of all the forces acting on it is 9 inches from the middle: what is the length of the rod ? *Ans.* $8\frac{1}{2}$ ft.

9. A beam, the weight of which is equivalent to a force of 10 lbs. acting at its middle point, is supported on two props at the end of the beam. If the length of beam be 5 ft.; find where a weight of 30 lbs. must be placed so that the pressure on the two props may be 15 lbs. and 25 lbs. respectively. *Ans.* 10 in. from the middle.

10. A horizontal straight bar, 6 ft. long and weighing 16 lbs., is supported at each end, and a weight of 48 lbs. is hung at 2 ft from one end: find the pressure upon each of the supports. *Ans.* 24 and 40 lbs.

11. Two parallel forces, P and Q, act at two points in a straight line 60 centimetres apart, in opposite directions. Their resultant is a force of one kilogramme acting at a point in the line 40 centimetres from the larger of the forces P and Q. Determine the values of P and Q. *Ans.* $1\frac{2}{3}$ and $\frac{2}{3}$ kilos.

12. A light rod 6 ft. long turns freely about a point 2 ft. from one end.

At that end a weight of 28 lbs. is suspended ; at the other end is suspended a weight of 12 lbs.; what weight must be suspended from the middle point of the rod to maintain equilibrium. *Ans.* 8 lbs.

13. A light rigid rod 19 ft. long turns freely about a point 9 ft. from one end at which a weight of 100 lbs. is suspended ; at the middle point is suspended a weight of 50 lbs. ; what weight must be suspended from the other end to maintain equilibrium ? *Ans.* 330 lbs.

14. A light pole 20 ft. long has suspended from its middle point a weight of 48 lbs., and rests on two walls 16 ft. apart, so that one foot of it projects over one wall and three feet over the other ; a weight of 96 lbs. is suspended from the end which projects furthest ; what is the pressure borne by each of the walls ? *Ans.* 3 lbs. and 141 lbs.

15. Three weights of 4, 5 and 7 lbs. are hung respectively at the centre and two extremities of a rigid bar 16 inches long. Find their resultant in magnitude and position.

First find resultant of 5 lbs. at A and 7 lbs. at B.

$$R = 5 + 7 = 12 \text{ lbs. acting at D,}$$

$$\text{so that} \quad 5 : 7 :: BD : AD$$

$$:: x : 16 - x$$

$$\therefore \quad 7x = 5\,(16 - x)$$

$$\therefore \quad x = 6\tfrac{2}{3}$$

$$\therefore \quad BD = 6\tfrac{2}{3}$$

$$\therefore \quad CD = 8 - 6\tfrac{2}{3} = 1\tfrac{1}{3}.$$

Now, find the resultant of 4 lbs. at C and 12 lbs. at D.

$$R = 4 + 12 = 16 \text{ lbs, acting at E,}$$

$$\text{so that} \quad 4 : 12 :: DE : EC$$

$$:: x : (1\tfrac{1}{3} - x)$$

$$12x = 4\,(1\tfrac{1}{3} - x)$$

$$\therefore \quad x = \tfrac{1}{3}$$

$$\therefore \quad DE = \tfrac{1}{3}$$

$$\therefore \quad EB = 6\tfrac{2}{3} + \tfrac{1}{3} = 7 \text{ in.;}$$

$\therefore$ the resultant of all the forces is 16 lbs. and acts at a distance of 1 inch from the force of 7 lbs.

16. Two weights of 3 ozs. and 5 ozs. hang at the ends of a rcad. 12 in. long and a third weight of 6 ozs. is placed 3 in. from the lighter weight. Find the position of the resultant. *Ans.* $6\frac{3}{7}$ in. from the heavier weight.

17. Find the centre of like parallel forces, 3, 2, 5, 7, lbs. which act at equal distances apart along a straight rod 12 in. long. *Ans.* $4\frac{4}{17}$ in. from the 7 lbs. weight.

EXAMPLES ON MOMENT.

1. A force of 15 lbs. acts on a body at a distance of 6 ft. from a point round which the body can turn. Find the moment of the force about the point.

Moment required $= 15 \times 6 = 90$. *Ans.* 90.

2. Replace a force of 16 lbs. which acts on an arm of 5 ft. by a force of 20 lbs.

Ans. 20 lbs. acting at an arm of 4 ft.

3. Find the moment of a force of 8 lbs. acting at an arm of 5 ft. If the moment remained the same and the arm increasd to 10 ft., what must be the magnitude of the force? *Ans.* 40; 4 lbs.

4. The moment of a force P about a point is 10 times the moment of another force Q about the same point, while the arm of P is five times the arm of Q; find the ratio of P to Q. *Ans.* 1 : 2.

5. A pole 12 ft. long sticks out horizontally from a vertical wall. It would break if a weight of 28 lbs. were hung at the end. How far out along the pole may a boy who weighs 8 stones venture with safety?

The effect of a force is measured by the product of the force and the distance at which it acts. (Vide Article 32).

$$\therefore \quad 28 \times 12 = 112 \times x,$$

$$\therefore \quad x = 3 \text{ ft.} \qquad \textit{Ans. } 3 \text{ ft.}$$

6. A uniform rod of length 20 ft. and weight 10 lbs. has a weight of 40 lbs. suspended from one extremity; what point in it must be supported that it may just balance?

Let the rod be AB and let D be the point about which it will just balance if supported, *i. e.* the point at which the resultant of 40 at A and 10 at C the middle point of AB (article 39) must act.

Now, AB = 20 ft, and AC = 10 ft.

Let AD = x ft.; then DC = $10 - x$.

Take moments round D, then we have (article 33)

$$40 \ AD = 10 \ DC$$

$$\therefore \quad 40x = 10(10 - x)$$

$$\therefore \quad 40x = 100 - 10x$$

$$\therefore \quad x = 2$$

The point of support must be 2 ft. from the end at which the weight is hung.

7. A light horizontal rod AB, 15 inches long, has a weight of 10 lbs. suspended at a point 6 in. from A. If the ends A and B are supported by vertical strings, find the tension of each string. *Ans.* 4 lbs. and 6 lbs.

8. Two men carry a block of stone weighing 120 lbs. on a light pole 12 ft. long; the stronger man bears 80 lbs. of the load. Find where the block must be suspended. *Ans.* 4 ft. from his shoulder.

9. A uniform rod 6 ft. long and weighing 5 lbs. is laid on a table with 6 in. projecting over the edge. What weight can be hung on the end of the rod before the rod will turn over? *Ans.* 25 lbs.

10. A rod AB whose length is 5 ft. and weight 30 lbs. is found to balance itself when supported on a prop 3 ft. from A. If this rod were placed horizontally on two points one under A and the other under B. what pressure would it exert on each point. *Ans.* 12 lbs. on A; 18 lbs. on B.

11. A uniform rod which is 4 ft. long and weighs 17 lbs. can turn freely about a point in it, and the rod is in equilibrium when a weight of 7 lbs. is hung at one end; how far from this end is the point which must be supported? *Ans.* 17 in.

12. A uniform rod of length $7\frac{1}{2}$ ft. turns freely about a point 1 ft. from one end and from that end a weight of 11 lbs. is suspended; what must be the weight of the rod that it may be in equilibrium? *Ans.* 4 lbs.

CHAPTER IV.

Work.

42. Work. How Measured?—When a force acts on a body, some effect is produced on the body and the force is said to have *done work.*

If the body acted on is moved in the *same sense* as the force, *the body is said to have done work* which is supposed to be *positive*; on the contrary if by the application of the force the body is moved in the sense opposite to that in which it was already moving, then work is said *to be done upon the body* and is reckoned *negative.*

Work, therefore, *implies two things*;—it implies that a body is moved through a certain distance and that force is required to move it. The work done is, hence, proportional to the distance through which the body is moved and to the magnitude of the force employed.

43. Work done against gravity.—We will here deal with only one kind of work, *viz.*, that done against gravity. To lift up a pound weight through one foot evidently an upward force of a pound acting through the distance of one foot will be required and the effort to raise a two-pound weight through a foot is twice as great as in raising one pound through the same height and again the exertion in raising two pounds through two feet will be twice as much as in raising the same weight through one foot only.

Generally, therefore, work is measured by the product of the weight into the distance through which it is raised; *i e.*, $W \times d$ is the **measure of work.**

Note that by the word "distance" is meant **effective distance**, that is, the distance measured in the same line as that in which the previously existing force is acting. Thus, if a man pulls a weight up a smooth incline 100 feet in length and 25 feet in height, though the actual distance traversed is 100 feet, the *effective distance* through which the weight is *pulled up* is 25 feet, because that is the distance through which the weight is raised against the direction of the force of gravity which is acting vertically downwards on it.

44. Unit of work.

Def.—A unit of work is the work done against gravity in lifting a certain weight through a certain height, such that **the product of the weight and the height is unity.**

On the English system of weights and measures the pound is the unit of weight and the foot is the unit of length, distance or height, and

1 lb. raised through 1 ft.,
or 2 lbs. raised through $\frac{1}{2}$ ft.,
or $\frac{1}{3}$ lb. raised through 3 ft.,
is called a **foot-pound** of work.

Again, on the French system, in which the kilogramme is the unit of weight and the metre is the unit of length or height,

1 kilogramme × 1 metre,
or 4 kilogrammes × ·25 metre,
or 0·5 kilogramme × 2 metres is called
a **kilogramme-metre** of work.

QUESTIONS.

1. Define *work*.

2. When is work said to be *positive* and when *negative* ?

3. How is work measured and why so ?

4. In estimating work done against gravity, why is the vertical height only taken into account ?

5. Define *unit of work*. What do you understand by the terms a *foot-pound* and *kilogram-metre* ?

EXAMPLES.

1. How many units of work are expended in lifting one ton one hundred feet high ?

The English unit of work is the foot-pound, *i. e.*, 1 lb. raised through 1 ft.

Now 1 ton = 2240 lbs.,

∴ 2240 lbs. raised through 100 ft. = 224,000 ft.-lbs. of work.

2. How much work does a man of 15 stone weight do against gravity in ascending a mountain 4,000 ft. high ?

The man has to raise himself, *i. e.*, a weight of 15 stones through a height of 4,000 ft.

Now 15 stones = 210 lbs.,

∴ 210 lbs. raised through 4000 ft. = 840,000 ft.-lbs. of work.

3. If a pit 10 ft. deep and with an area of 4 sq. ft. be excavated and the earth thrown up, how much work will have been done, supposing a cubic foot of earth to weigh 90 lbs.?

The cubic contents of the pit is $(10 \times 4 =)$ 40 cubic feet, and the weight of the whole mass of the earth is $90 \times 40 = 3600$ lbs. Now the upper most layer is lifted through hardly any height, and the lowermost layer through exactly ten feet, the layer a foot below the surface through one foot, another nine feet below through nine feet, and so on. The average height through which the whole mass is lifted up is therefore five feet and the total work done is $3{,}600 \times 5 = 18{,}000$ ft.-lbs.

4. In building a wall, 5 tons of materials are used, and the height to which the wall is raised is 12 ft. Find the amount of work done in building it.

All the material is not raised to the height of 12 ft., nor is it all spread on the ground, but while the lowermost layer is level with the ground, the uppermost is twelve feet above the ground; the whole mass is therefore raised to an average height of 6 feet.

Now 5 tons $= 2{,}240 \times 5 = 11{,}200$ lbs.

∴ the work done is $11{,}200 \times 6 = 67{,}200$ ft.-lbs.

5. Find the work expended in pulling a body, weighing 50 kilogrammes, 250 metres up an incline that rises 1 in 10.

As the slope is 1 in 10, therefore on reaching the top of the slope the vertical height attained is 25 metres, and the weight is really raised through that height against gravity (*vide* Art. 43). Hence the work done is $50 \times 25 = 1{,}250$ kilogram-metres.

6. Convert a foot-pound into kilogram-metre and *vice versa*. *Ans.* 0·137 kilogram-metre and 7·23 ft. lbs.

7. A cooly carries a weight of one maund up a hill 2,500 feet high ; the weight of the cooly himself is 5 maunds 8 lbs.; what is the total amount of work done ? *Ans.* 440,000 ft.-lbs.

8. If in the above example, the time taken to reach the top is 3 hours, find the rate of work. *Ans.* 2,444·4 ft.-lbs. per minute.

9. If the average rate of work done by a man is 15 foot-pounds per second, calculate the number of tons that two men can raise in three hours from the bottom of a pit 30 feet deep. *Ans.* $4\frac{23}{28}$ tons.

10. In rowing the work done per minute is estimated at 4,000 ft.-lbs.; find how long will four oarsmen take to propel a boat, whose resistance is one-eighth of a ton, a distance of two miles. *Ans.* 3h. 4m. 48s.

11. What work per hour will a horse perform travelling at the rate of $3\frac{1}{2}$ miles per hour, supposing the force of traction to be 100 lbs.? *Ans.* 1,848,000 ft.-lbs.

12. An engine of 20 horse-power saws 150 square feet of teak in 6 minutes; how many units of work are consumed in cutting one square foot? A horse-power is equivalent to 550 ft.-lbs. per second. *Ans.* 400 ft.-lbs.

13. In what time will an engine of 80 H.-P. raise 10 tons from a depth of 100 ft.? *Ans.* 2·26 minutes.

14. A well, whose section is 16 sq. ft. and whose depth is 300 ft., is full of water; find the work done in pumping the water to the level of the top of the well. *Ans.* 45 million foot-pounds.

15. In the above example determine the H. P. of the engine which would accomplish the work in one hour. *Ans.* $22\frac{8}{11}$ H. P.

16. The surface of water in a well is at a depth of 20 ft. and when 300 gallons have been pumped out the surface is lowered to 24 ft. If a gallon of water weighs 10 lbs. find the amount of work done. *Ans.* 66000 ft.-lbs.

17. A body weighing 3 lbs. is drawn 40 ft. up an incline which rises 3 ft. in height for every 5 ft. along the incline. Find the work done. *Ans.* 72 ft.-lbs.

PART II.—THE MECHANICAL POWERS.

CHAPTER I.

Machines

44. Machines. A body on which a force is imprssed and which is therefore capable of doing work, is said to possess *energy*. Instruments or contrivances by which the amount or direction of a force applied to a body can be altered or by which energy is transferred from one point to another, are called *machines*.

By the help of a machine

(i) we may *change the amount* of a force, as when a man working at a crane lifts up a very heavy weight ;

or (ii) we may *change the direction* of the force applied in a manner convenient to us, as when we use a windmill to work a pump ;

or lastly, (iii) we may *change the rate* at which the force is acting, as in the case of a bicycle.

Def. The power of doing work is called **energy.**

Def. A **machine** is an instrument for **transferring energy** in such a manner that requisite **useful** or **desirable work** is done.

45. Principle of Energy. Equation of work.— It has been found as the result of innumerable experiments on the subject of perpetual motion and others, that it is as **impossible to create energy** as it is to create matter ; and whenever energy is spent in producing work, it is always at the expense of some other equivalent quantity of energy

previously existing ; it is therefore, *impossible for us to construct a machine which shall do work of itself* (*i.e.*, produce energy) without consuming at least an equal quantity of pre-existing energy, and from no machine can we obtain a greater amount of work than is applied to it. This is the great principle of **Conservation of Energy.**

This principle of energy, applied to the ordinary cases of the mechanical powers, means that to obtain with the agency of any one of these powers a certain number of units of work same amount of work must be spent. In other words, the foot-pounds (or kilogramme-metres) of work obtained cannot exceed the foot-pounds (or kilogramme-metres) of work spent; it is as a rule less owing to resistance to free motion in the several parts of the machine ; this resistance is called *friction.*

If P be the force or '*power*' acting through an effective distance d, the utmost it can do is to overcome another force or '*weight*' W through an effective distance d', such that $\mathbf{P \times d = W \times d'}$. This expression is called the **equation of work.**

In what follows we will consider our machines to be frictionless, and therefore the equation of work will always hold good.

Def. The force or effort applied at one end of a machine is called **power.**

Def. The opposing force or resistance overcome by the power at the other end of the machine is called **weight.**

46. Virtual work—As $P \times d = W \times d'$, therefore $P \times d - W \times d' = 0$; hence the **total** or algebraical sum of the **work done** in a machine **by all the forces** acting on it is **zero.** As in Statics we only consider cases in which

no motion takes place by the action of the forces, therefore in a machine, statically considered, the displacements **d** and **d′** do not actually take place ; they are only *imaginary* or '*virtual*,' and the above principle of work as applied to statical forces is, hence, called the principle of **virtual work** or of virtual velocities.

47. Simple Machines or Mechanical Powers.—Every machine such as a crane, a windmill-pump or a bicycle, consists of several *simple parts*, every one of which can be classified under the following six main heads ; these are called **simple machines** or **mechanical powers.**

They are so called because they are generally **employed for enabling a small power to support a large weight.**

Def. The simplest forms of machines are called **mechanical powers.**

The mechanical powers are **six** in number.

1. The **Lever,**
2. The **Wheel and axle,**

} in which a solid body is moveable about a fixed point.

3. The **pulley**, in which is employed a flexible string.
4. The **Inclined plane,**
5. The **Wedge,**
6. The **Screw,**

} which are formed of hard inclined surfaces.

48. Mechanical Advantage.—On the 'principle of energy', the general condition of equilibrium in a machine is that P × *the distance through which P moves*

= W × *the distance through which W moves* ;

i.e., $\dfrac{\mathbf{W}}{\mathbf{P}} = \dfrac{\textit{Power-distance}}{\textit{Weight-distance}}$.

Def. When two forces, a 'power' and a 'weight,' act on an machine and maintain it i equilibrium, the ratio of the weight to the power $\left(\frac{W}{P}\right)$ is called the **modulus** of the machine.

When the value $\frac{W}{P}$ is **greater than unity**, the machine is said to work at a **mechanical advantage.** The reason for this is evident, because the object of a machine is usually to balance a large 'weight' with a small 'power,' and when this end is attained, we gain what we want or we get a certain kind of advantage out of the machine.

On the other hand, when the value of $\frac{W}{P}$ is **less than unity**, the machine is said to act at a **mechanical disadvantage.***

There is a **limit to the mechanical advantage** that could be gained through the means of a machine. Because as $\frac{W}{P} = \frac{\text{Power-distance}}{\text{Weight-distance}}$, mechanical advantage will depend on the latter ratio, which is a maximum when the 'power-distance' is the greatest possible and the 'weight-distance' the least possible ; in every machine we can neither increase the power-distance nor decrease the weight-distance beyond certain limits, therefore in every such case there is a maximum value of $\frac{W}{P}$ which can never be exceeded.

* The practice of calling the *ratio* $\frac{W}{P}$, itself, *mechanical advantage*, irrespective of its numerical value being greater or less than unity is more and more coming into vogue. Therefore when the weight or resistance overcome is greater than the power or the force applied, the '*advantage*' is said to be greater than unity or 'positive,' and when the weight is smaller than the power, the '*advantage*' is said to be less than unity or 'negative.,

QUESTIONS.

1. Define the term *Energy*. and illustrate it by examples.

2. What is a *Machine*? What functions can a machine perform?

3. What do you understand by the term *Conservation of Energy*?

4. Why is 'perpetual motion' considered an impossibility?

5. Explain the meaning of the terms *Power* and *Weight* as applied to machines.

6. Give the *equation of work* and indicate its importance in Dynamics

7. When is work said to be *virtual* and why?

8. What machines are called *simple* and why? Why are simple machines often called mechanical *powers*?

9. Point out that some of the six 'simple machines' are merely modified forms of others.

10. Explain the terms *modulus, mechanical advantage* and *mechanical disadvantage*.

11. Is there any limit to the mechanical advantage that may be secured with a machine of a given type? Give reasons.

CHAPTER II.

The Lever.

49. The lever, straight and bent. Kinds of Levers.

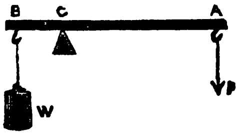
Def. The **lever** is a rigid body capable of turning round a fixed point or axis.

This fixed point is called the **fulcrum**, and the two parts into which the body is divided by the fulcrum are called the **arms** of the lever.

When the arms of a lever are in a straight line, it is called a **straight lever**; when they are not so, it is said to be **bent.** We will only consider levers of the first sort and consider them to be without weight, *i.e.*, *light*, unless otherwise specified.

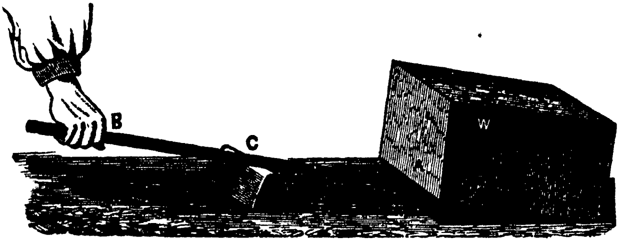
Kinds of Levers.—Levers are divided into **three** classes or kinds, **according to the position of the fulcrum** with respect to the power and the weight.

50. Lever of the first kind.—In levers of the **first kind**, the **power and weight** are applied on **opposite sides of the fulcrum** (C), and **act** in the **same direction.**



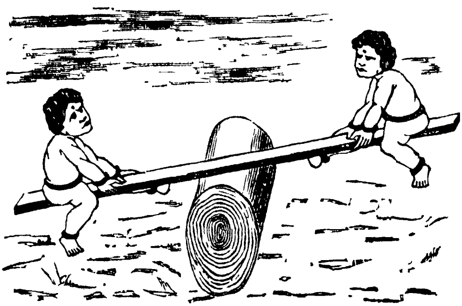
EXAMPLES.—A common **balance,** in which the **point of suspension** is the **fulcrum** and the **weight and power** are the **load and counterpoise**; the **ladle** as used by confectioners in stirring milk, in which the point of support or **fulcrum** is the **edge of the pan, W** is the **milk** at one end of the ladle, and **P** is the **force** applied by the man at the other. A **poker** used to stir the fire, in which the

bar of the grate is the **fulcrum, W** is the **coal,** and **P** is the **force** applied by the hand ; the **crow-bar** when



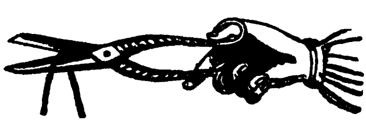
employed to lift up a **weight placed at one end** (*as in the figure*) with **P** at the other and the fulcrum between as at **C.**

The **see-saw** is also a lever of this class, the **block** on

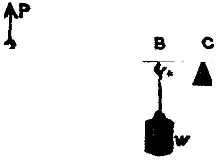


which the plank is supported is the **fulcrum,** while the weights of the two **boys** at the two ends represent the **power** and the **weight.**

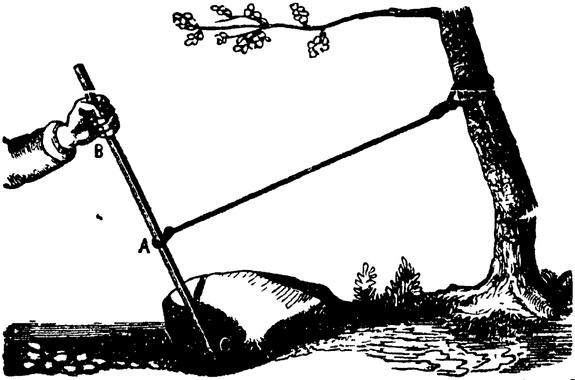
Again, a pair of **scissors** is a **double lever** of the first kind, the **rivet** being the common **fulcrum of** the two levers ; on one side of the fulcrum is the '**weight,**' viz., the **resistance of the material** to be cut, and on **the other side** is the **power** applied by the fingers.



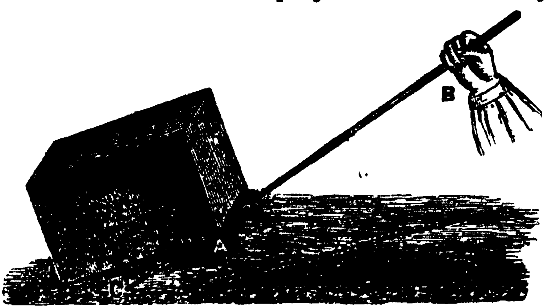
51. Lever of the second kind.—In levers of the **second kind, P** and **W** are applied on the **same side of the fulcrum** and **act** in **opposite** directions, the **power** being applied at a **greater distance from the fulcrum** than the weight is.



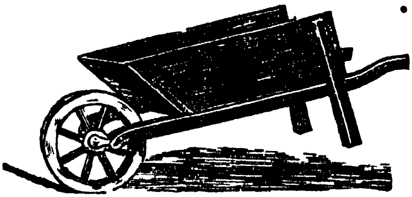
EXAMPLES.—The **crowbar** (*as used in the figure*), to bring down a tree, in which the **fulcrum** is the point of support **C, W** is the **resistance offered by the tree** acting at **A** through the rope, and **P** is the muscular **force** applied in the **opposite direction at B further away from C** than A is.



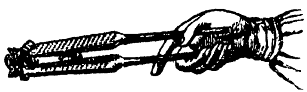
The crowbar when *employed to roll a weight* is also a lever of the second kind, as the **weight acts between the power and the fulcrum**



and in a direction opposite to **that** of the power. The



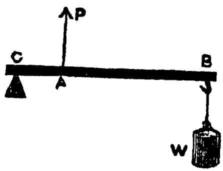
wheel-barrow is another instance in point, the part of the wheel touching the ground is the **fulcrum, P** is applied to the **handles** at the other end to keep the barrow lifted up, while the **weight** is supported **between.** An English **nut-**



cracker or an Indian **betel-nut cutter** is a **double-lever** of the second kind, in which the **joint at one end** is the **fulcrum**, the **power** is applied by the hand **at the other end,** and the **weight** is the **resistance** offered by the **nut** which is **placed midway.**

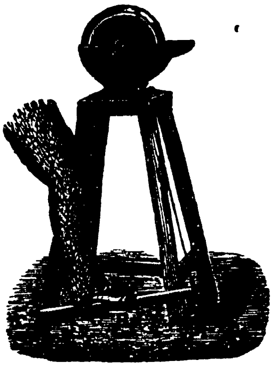
The **oar** of a boat is also a lever of the second kind. The **blade** of the oar **resting** at each stroke against the water (which is assumed to be stationary for the time being) is the **fulcrum,** the **power** is applied at the **other end** to the **handle** of the oar and the **resistance** of the **rowlock midway** is W.

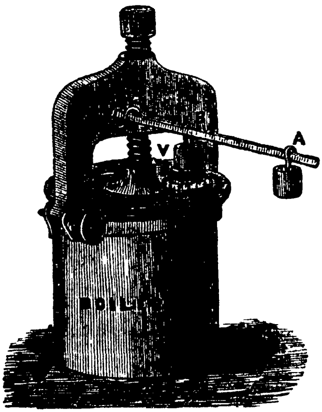
52. **Lever of the third kind.**—In levers of the **third**



kind, P and W are applied on the **same side of the fulcrum** and **act** in **opposite** directions, the **power** being **near to the fulcrum** than the weight is.

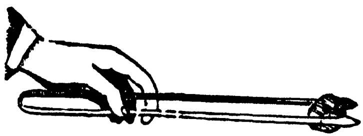
EXAMPLES :—The **treadle** of a grinding stone or of a sewing-machine is a lever of this class, in which **C**, the point on which the treadle oscillates, is the **fulcrum**, **P** is the muscular force applied **midway (at B)** by the foot, and **W** is the **resistance** experienced at A in turning the grind-stone or the sewing-machine; the **lever of a safety-valve** of a boiler, in which **W** is the load placed **at the extremity** of the lever at A, and **P** is the **upward force of the steam** acting through the valve at B, is also of the third-kind. The human **fore-arm** is also an instance of the lever of the same class; a strong **muscle** passes down in front of the joint of the elbow and is inserted into the one of the two parallel bones which compose the framework of the fore-arm; on being contracted at the impulse of the will, the force of the **muscle (P)** elevates the **hand** with

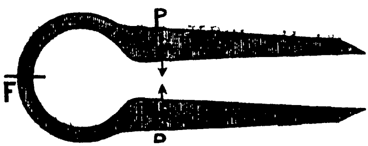




any load that may be held in it (**W**), bending the arm upon the **elbowjoint** at the other end, which is the **fulcrum.**

A pair of **tongs** is a **double-lever** of this kind, in which the **joint** is the common **fulcrum at one end**, **W** is the weight held **at the other end**, and **P** is the power applied **midway** by the hand ; the **shears** used for cutting grass is another instance of a double-lever of the third kind ; the **fulcrum** is the point at one end where the two blades meet, **W** is the resistance offered by the **grass**, and **P** is applied midway **by the hand** on the blades.





53. Condition of equilibrium.—*The moment of P round the fulcrum is equal to the moment of W round the fulcrum.* This is called the **principle of the lever.** We will now proceed to prove it separately for the three kinds.

First kind.—(*Vide* first figure, art. 50). As the lever rests on the fulcrum C, the combined effect of the two *parallel* forces P and W is felt at that point equal to P + W (article 27), and as the fulcrum resists or opposes this latter pressure, the *re-action* (R) at C is equal and opposite to it. Hence there are really *three parallel forces in equilibrium* acting on it, *viz.*, P at A, W at B, and R at C, the direction of the latter being opposite to that of the other two. Now (by article 34) *the algebraical sum of*

the moments of these forces must be equal to zero ; therefore taking moments round the point C and assigning to them their proper signs (article 32), we get

$$W \times BC + R \times 0 - P \times AC = 0$$

$$\therefore P \times AC = W \times BC,$$

$$\text{hence} \quad \frac{W}{P} = \frac{AC}{BC}.$$

As C may be anywhere between A and B, AC may be greater than, equal to or less than BC, and consequently W may also be greater than, equal to or less than P ; the first kind of lever thus **may or may not act at a mechanical advantage.**

The reaction (R) is equal to the resultant of P and W, *i. e.*, to P + W. Hence **R = P + W.**

Second kind.—(*Vide* first figure, art 51.) Here again the re-action (R) at the fulcrum C is equal to the resultant of P and W, and as before there are three parallel forces in, equilibrium acting on it and taking moments round C, we find

$$-P \times AC + W \times BC + R \times 0 = 0$$

$$P \times AC = W \times BC,$$

$$\text{hence} \quad \frac{W}{P} = \frac{AC}{BC}.$$

As in this case, AC is *always greater* than BC, $\frac{W}{P}$ is always greater than unity, and the lever, therefore, **al ways works at an advantage.**

As *W is always greater than* ***P*** *and in opposite direction,* therefore the resultant of P and W is equal to W — P (article 28), and the re-action is also equal to it ; *i. e.*, **R = W — P.**

Third kind.—(*Vide* first figure, art. 52). In the third kind, taking moments with proper signs as before, we get

$$R \times O + P \times AC - W \times BC = O$$

$$\therefore\ P + AC = W \times BC,$$

$$\text{hence } \frac{W}{P} = \frac{AC}{BC}.$$

In this system, AC is always less than BC and therefore $\frac{W}{P}$ is always *less than* unity and the lever **always acts at a disadvantage.**

As *W is always less than* P, therefore the resultant of P and W is equal to P — W (article 28), and the re-action is also equal to it, *i. e.*, **R = P — W.**

54. Recapitulation.—In all the three systems P × AC (power arm) = W × BC (weight arm), *i. e.*, the moment of P = the moment of W about the fulcrum.

In the first system. R = P + W.
In the second system, R = W — P.
In the third system, R = P — W.

In the *first* system, we gain *mechanical advantage or not* as the power arm is longer or shorter than the weight arm ; in the *second*, there is *always mechanical advantage*; while the *third* acts *always at a disadvantage.*

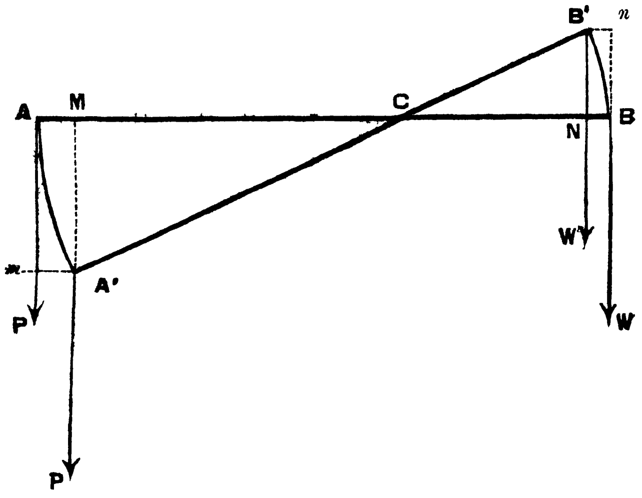
Position of the fulcrum.—We must observe that in the first system as the re-action is equal to the sum of P and W, it is opposite in direction to P and W, and as the fulcrum causes the re-action, it must be underneath the lever in order to exert an upward pressure when the power and the weight act downwards ; in the second system as W is equal to R and P together, R must act in the same direction as P, *i. e.*, upwards, and the fulcrum must be underneath the lever; and finally in the third system as P

is equal to R plus W, R must act in the same direction as W, *i. e.*, downwards, and therefore the fulcrum must be fixed above the lever.

55. Application of the principle of work.

On the *principle of energy*, in a lever the equation of work is

P × effective distance through which P moves = W × effective distance through which W moves (article 45).



Now let AB be a lever turning on C and let P and W be applied at the two extremities A and B respectively. Suppose the lever is turned through the angle ACA′ = angle BCB′, then the equation of work is

$$P \times Am = W \times Bn$$

or

$$P \times A'M = W \times B'N.$$

$$\therefore \frac{W}{P} = \frac{A'M}{B'N} = \frac{A'C}{B'C} = \frac{AC}{BC};$$

$$\text{hence } P \times CA = W \times CB,$$

$$P \times \text{its arm} = W \times \text{its arm},$$

$$\textit{i.e.}, \text{ moment of } P = \text{moment of } W.$$

EXAMPLES OF LEVER OF THE FIRST KIND.

1. In the first kind of lever the power is 20 lbs., the arm of P equal to 3 ft. and the arm of W equal to 3 inches. Find W.

$$P \times P\text{-arm} = W \times W\text{-arm}$$

$$\therefore \quad 20 \times 36 = W \times 3$$

$$\text{and} \quad W = \frac{20 \times 36}{3} = 240 \text{ lbs.}$$

Ans. 240 lbs.

2. Two bars, uniform in length and weight, are suspended at distances 2 ft. and 5 ft. respectively, from the fulcrum of a lever on opposite sides of it and just keep the lever at rest. Compare the weights of these two bars. *Ans.* 5 : 2.

3. In the handle of a common pump, the piston rod is placed at a distance of 3 inches from the fulcrum, a power of 60 lbs. is applied at the distance of 30 inches from the fulcrum. What will be the force with which the piston will be lifted? *Ans.* 600 lbs.

4. A lever with the fulcrum between the power and the weight has its arms 5 ft. and 10 ft. respectively; if a force of 20 lbs. acts at the extremity of the longer arm, what weight can the lever support at the other end? *Ans.* 40 lbs.

5. Two weights of 3 lbs. and 7 lbs., respectively, hung from the extremities of a lever 1 yard long equilibrate each other; find the position of the fulcrum.

(See figure 1, art 50.)

The lever is a straight one of the first kind, and, for equilibrium, P × its arm = W × its arm.

$$P = 3 \text{ lbs.}, \; W = 7 \text{ lbs.}$$

$$\text{Power-arm} = CA = x \text{ yards.}$$

$$\text{Weight-arm} = CB = AB - CA = (1 - x) \text{ yards.}$$

Therefore, by substitution,

$$3 \times x = 7(1 - x)$$

$$\text{and } 10x = 7$$

$$\text{hence, } x = 0{\cdot}7$$

The power-arm CA is, therefore, 0·7 of a yard and the fulcrum C is at a distance of 0·7 yard from the point A where P is applied.

6. If the shorter arm of a lever of the first kind be 8 inches, what is the length of the lever, if P be 48 when W is 312?

As before, by substitution,

$$48\,x = 312 \times 8 = 2496$$
$$\therefore\ x = 52 = CA,$$
$$\therefore AB = CA + CB,$$
$$= 52 + 8 = 60.$$

Hence, the length of the lever is 60 inches = 5 ft.

7. Two weights of 6 lbs. and 8 lbs. are hung from the ends of a lever 7 ft. long; where must the fulcrum be placed so that they may balance? *Ans.* 3 ft. from the larger weight.

8. Two boys play at see-saw upon a plank 12 feet long; they weigh 80 lbs. and 120 lbs. Where will the fulcrum be? *Ans.* 4·8 feet from the end where the big boy is.

9. Two weights of 3 and 4 lbs. respectively balance at the extremities of a lever 14 feet long; find the position of the fulcrum. *Ans.* 8 feet from the 3 lbs. weight.

10. A lever is 18 in. long; where must the fulcrum be placed in order that a weight of 6 lbs. at one end may balance double its weight at the other end? *Ans.* 12 in. from the smaller weight.

11. A crowbar 10½ ft. long is used to raise a weight of 500 lbs. It rests upon a stone 1 foot from the end. A boy pulls with a force of 50 lbs. at the other end. Will he succeed in raising the weight? *Ans.* No.

12. Two weights, which together weigh 6½ units, are hung at the ends of a straight lever and balance. If the fulcrum is four times as far from one end as from the other, find each weight. *Ans.* 1·3 and 5·2 units.

13. A beam, the length of which is 4·5 metres balances at a point 1·5 metre from one end. If to this end a weight of 40 kilogrammes be hung find the least force which applied at the other end will support this weight. *Ans.* 20 kilogs.

14. The arms of a lever are as 2 : 3 and the pressure on the fulcrum is 35 lbs.; find the two weights. *Ans.* 14 lbs. and 21 lbs.

15. A man, who weighs 5 maunds, wishing to raise a rock, rides the end of a crowbar 5 feet long, which is propped at the distance of 5 inches from the end in contact with the rock. What is the pressure on the prop?

The lever is of the first kind, and, for equilibrium, P $\times$ its arm $=$ W $\times$ its arm;

$$\therefore 5 \times 55 = W \times 5,$$
$$\text{hence } W = 55 \text{ maunds.}$$

Again, by the 'principle of parallel forces' (articles 27 and 53) a a lever, the pressure on the prop or fulcrum is the algebraical sum of the power and the weight; therefore, the pressure is one of 60 maunds.

16. A man who weighs 160 lbs., wishing to raise a rock, leans with his whole weight on one end of a horizontal crowbar five feet long, which is propped at the distance of four inches from the end in contact with the rock. What force does he exert on the rock, and what pressure has the prop to sustain? *Ans.* 2,240 and 2,400 lbs. respectively.

17. If A be that arm of a straight lever at which the power acts and B that arm at which the weight acts, show that mechanical advantage is gained or not by the use of the lever, according as A is greater or less than B.

When W is greater than P, the lever is said to work at a mechanical advantage; when less, at a disadvantage (*vide* art. 48).

Now by the principle of the lever, for equilibrium,

$$P \times A = W \times B,$$
$$\text{and } \frac{W}{P} = \frac{A}{B};$$

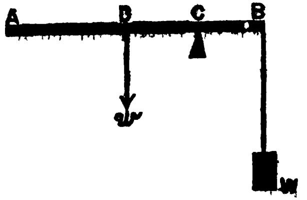
therefore, if A is greater than B, W is greater than P, and the lever acts at a mechanical *advantage*; conversely if A is less than B, W is less than P, and the lever works at a *disadvantage.*

18. Find the mechanical advantage of a straight lever whose arms are 2 ft. and 6 ft. *Ans.* 3.

19. If the mechanical advantage of a lever of 12 ft. be three, what are the lengths of the arms? Ans 3 ft. and 9 ft.

20. Suppose a uniform rod, 20 feet in length and 10 lbs. in weight, is supported on a fulcrum at the distance of two feet from the extermity at which a weight is hung; required a weight, which, hanging at that extremity, will bring the rod to a horizontal position.

The weight of a beam is supposed to act at its centre of gravity and the centre of gravity of a uniform beam is at its centre; therefore, we may imagine the whole weight of the beam, viz., 10 lbs., to act at the middle point (D) of the beam.



Then, CD is the distance from the fulcrum at which this weight of 10 lbs. acts, and this weight is kept in equilibrium by a weight (W) suspended at B.

Therefore, by the 'principle of the lever.'

$$10 \times CD = W \times CB.$$

Now CB = 2 feet, and DB being half of AB is 10 ft.,

$$\therefore CD = DB - CB$$
$$= 10 - 2 = 8$$
$$\therefore 10 \times 8 = W \times 2$$
$$80 = 2W$$
$$\therefore W = 40,$$

i.e., the weight necessary to keep the beam horizontal is 40 lbs.

21. A straight lever 8 ft. in length can turn about a fulcrum distant 2 ft. from one of its end. What weight must be attached to the end of the longer arm to balance a weight of 15 lbs. attached to the end of the shorter arm? *Ans.* 5 lbs.

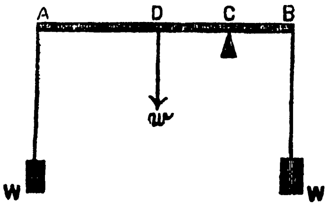
22. A straight lever 2 ft. long balances about its middle point when unloaded, and with a 5 lbs. weight at one end it balances about a point 4 inches from that end, find the weight of the lever. *Ans.* $2\frac{1}{2}$ lbs.

23. A uniform rod which is 12 ft. long and which weighs 5 lbs. can turn freely about a point in it, and the rod is in equilibrium when a weight of 1 lb. is hung at one end. Find the position of the fulcrum. *Ans.* 5 ft. from 1lb.

24. A uniform bar of iron 10 ft. long projects 6 ft. over the edge of a wharf, there being a weight placed upon the other end; it is found that when this is diminished to 3 cwt. the bar is just on the point of falling over. Find its weight. *Ans.* 12 cwts.

25. A uniform rod, 2 ft. long, weighs 1 lb. What weight must be hung at one end in order that the rod may balance on a point 3 inches from the end? *Ans.* 3 lbs.

26. A lever 10 ins. long, the weight of which is 4 ozs., and acts at its middle point, balances about a certain point when a weight of 6 ozs. is hung from one end; find the point. *Ans.* 2 ins. from the 6-oz. weight.



27. A uniform beam 18 feet long rests in equilibrium upon a fulcrum 2 feet from one end, having a weigh of 5 lbs. at the end furthest from the fulcrum, and one of 110 lbs. at the other; find the weight of the beam.

The weight of a beam is supposed to act at its centre of gravity, and the centre of gravity of a uniform beam is at its centre (*vide* article 39), therefore, the weight of the beam (w) may be supposed to act at D, the central point.

Then taking moments, P $\times$ its arm + w $\times$ its arm = W $\times$ its arm.

$$\text{Now P} = 5 \text{ lbs., W} = 110 \text{ lbs.,}$$
$$\text{power-arm} = CA = 16 \text{ ft.,}$$
$$\text{and weight-arm} = CB = 2 \text{ ft.,}$$
$$\text{and arm of } w = CD = DB - CB = 9 - 2 \text{ or } 7 \text{ ft.;}$$

therefore, by substitution,

$$5 \times 16 + w \times 7 = 110 \times 2$$
$$\therefore 7w = 140$$
$$\therefore w = 20$$

i.e., the weight of the beam is 20 lbs.

28. A lever is 5 metres long, and its weight is 5 kilogrammes, and acts at the middle point. The resistance to be overcome at one end is 10 kilogrammes and the force applied at the other end is 5 kilogrmmes: find the position of the fulcrum. *Ans.* $1\frac{1}{3}$ metre from the W-end.

29. A straight lever 20 inches long weighs 15 ozs. Where must the fulcrum be placed in order that the lever may be in equilibrium when a weight of 16 ozs. is hung at one end, and a weight to 9 ozs. at the other? *Ans.* $8\frac{1}{4}$ in from the 16 oz. weight.

30. What is the pressure on the fulcrum when the lever is in equilibrium? *Ans.* 40 ozs.

31. A uniform beam, whose weight is 50 kilogrammes and length 5 metres, has weights of 20 kilogrammes and 30 kilogrammes hung on its ends. Find the point on which the system will balance. *Ans.* 2·25 metres from 30 kilogrammes.

32. A uniform straight lever 3 ft. long weighs 4 lbs. What weight on the shorter arm will balance 10 lbs. on the longer, the fulcrum being 1 foot from the end? *Ans.* 22 lbs.

33. The two arms of a lever, of uniform thickness and density are 8 in. and 4 in., respectively; a weight of 16 seers is suspended from the shorter arm. What power will support it, the weight of the lever being 12 seers? *Ans.* 5 seers.

34. A lever weighs 5 kilogrammes, and its weight acts at its middle point, the ratio of its arms, is 1 : 3. If a weight of 80 kilogrammes be hung from the end of the shorter arm, what weight must be suspended from the other end to prevent motion? *Ans.* 25 kilos.

35. A heavy uniform beam, 10 ft. long, whose mass is 10 lbs., is supported at a point 4 ft. from one end; at this end a mass of 6 lbs. is placed. Find the mass required at the other end to balance the beam. *Ans.* $2\frac{1}{3}$ lbs.

36. If a bar whose length is 21 and weight W be used as a lever of the first kind, find the distance of the fulcrum from the centre of the bar, when a power P sustains a weight W. *Ans.* $\frac{(W-P)\,l}{P+W+W}$.

37. Two weights of 4 lbs. and 8 lbs. balance when suspended from the ends of a straight lever with the fulcrum 1 ft. from the larger weight; when P lbs. are added to each weight, the fulcrum has to be shifted to a further distance of 2 inches. Find the value of P and the length of the lever.

Let x be the length of the power-arm, then, as the lever is of the first kind,

$$4\,x = 8 \times 1$$
$$x = 2 \text{ ft.},$$

the length of the lever is 3 feet.

When equal weights, each P lbs., are added on both sides, the new condition of equilibrium is

$$(4 + \text{P}) \times 1\tfrac{5}{6} = (8 + \text{P}) \times 1\tfrac{1}{6}$$
$$8\,\text{P} = 24$$
$$\text{and P} = 3 \text{ lbs.}$$

38. Two weights of 12 lbs. and 8 lbs. balance at the end of a lever 10 ft. long; find how far the fulcrum ought to be moved for the weights to balance when each is increaseed by 2 lbs.

Ans. 2 inches on to the side of the smaller weight.

39. A uniform rod AB, weighing 10 lbs. and 10 ft. in length is found to balance about a point 8 ft. distant from A when weighted at the other end. A weight of 6 lbs. is next fastened at A; about what point will the rod now balance?

(*Vide* figure of Example 20.)

The weight of the beam acts at the middle of its length, as it is uniform (article 39); therefore, taking moments round C, we get

$$10 \times CD = W \times CB.$$

$AC = 8$ ft., and $AD = 5$ ft., therefore, $CD = 3$ ft.,

and $CB = 2$ ft.;

therefore, by substitution,

$$10 \times 3 = W \times 2$$

and $W = 15$ lbs.

When a weight of 6 lbs. is next fastened at the end A, the new condition of equilibrium is

$$6 \times x + 10\,(x-5) = 15\,(10-x)$$

where x is the distance of the point of equilibrium from A.

Hence $31x = 200$

$x = 6\frac{14}{31}$ ft. from A.

40. A lever, 10 ft. long, balances about a point 4 ft. from one end; but, when loaded with 20 lbs. at one end and 4 lbs. at the other, it balances about a point 3 ft. from the end. Find the weight of the lever. *Ans.* 32 lbs.

41. A lever, AB, 12 ft. long, balances about a point 1 foot from A, when a weight of 13 lbs. is suspended from A; it will also balance about a point 1 foot from B, when a weight of 11 lbs. is suspended from B. Show that the centre of gravity of the lever is 5 inches from the middle point.

Let x be the distance between the fulcrum in the first position and the centre of gravity of the lever and let w be the weight of the lever. Then, the lever being of the first order, for equilibrium

$$13 \times 1 = w \times x; \qquad \ldots \text{(i)}$$

when the weight of 11 lbs. is next hung from B, the distance between the new point of equilibrium and the centre of gravity is

$$[12-1-(1+x)] \text{ feet} = [10-x] \text{ feet},$$

and for equilibrium,

$$11 \times 1 = w\,(10-x) \qquad \ldots \text{(ii)}$$

Eliminating x, we get

$$24 = 10\,w$$
$$\text{and } w = 2\tfrac{2}{5} \text{ lbs.}$$

Substituting this value of w in the first equation, we obtain

$$13 = 2\tfrac{2}{5}x,$$
$$\text{and } x = 5 \text{ ft. } 5 \text{ in.};$$

and the distance of the centre of gravity from the point A is 6 ft. 5 in. *i.e.* 5 inches from the middle point of the lever.

42. A beam AB, 20 ft. long and weighing 10 lbs. rests on two props, one placed 2 ft. from A and the other 3 ft. from B. Find the pressures on the props when a weight of 40 lbs. is suspended from A and a weight of 50 lbs. from B. *Ans.* 40 lbs. on A and 60 lbs. on B.

43. A uniform bar weighs 2.6 grammes per centimetre. Find its length when a weight of 12·5 grammes suspended at one end keeps it in equilibrium about a fulcrum 12 centimetres from the other end. *Ans.* 20 cm.

44. A straight lever AB, 40 inches long, balances about a fulcrum, 12 in. from A, when 45 lbs. is hung at A, 20 lbs. 9 in. from A, 6 lbs. 4 in from B, and 12 lbs. at B. Find the weight of the lever. *Ans.* 15 lbs.

45. Two weights, hanging at the ends of a straight lever, keep it in equilibrium. The one exceeds the other by 25 lbs. and the longer arm exceeds the shorter by 5ft. If the pressure on the prop be 65 lbs., find the weights and the length of the lever. *Ans.* 20 lbs., 45 lbs.; 13 ft.

46. A man carries a bundle at the end of a stick over his shoulder; show that as the portion of the stick between his shoulder and his hand is shortened, the pressure on his shoulder is increased. Does this change alter the pressure on the ground? Give the reason.

By the principle of 'parallel forces,' in a lever the pressure on the fulcrum is the algebraical sum of the power and the weight (*vide* arts. 53 and 54).

Again, for equilibrium, P $\times$ its arm = W $\times$ its arm, and if "the portion of the stick between the shoulder and the hand is shortened," the power-arm is shortened and the weight-arm is lengthened at the same time, and to maintain equilibrium P will have to be increased; also as the pressure on the shoulder, which acts as the fulcrum, is equal to the sum of the power and the weight, it increases by the shortening of the power-arm to the extent to which P is increased.

This, however, does not alter the pressure on the ground, because the power is the muscular exertion of the arm which does not add to the weight of the man or of the stick.

47. Two forces P and Q acting at the ends of a lever keep it at rest. When A is trebled and P is increased by 6 lbs., the lever is again horizontal. Find P. *Ans* 3 lbs.

EXAMPLES OF LEVER OF THE SECOND KIND.

1. What would be the position of a weight of 20 lbs. on a lever in order that a power of 2 lbs. at one extremity may just raise it, the fulcrum being at the other extremity, and the length of the lever 10 feet.

Vide first figure art. 51.

The lever is evidently of the second kind, and, for equilibrium,

$$P \times \text{its arm} = W \times \text{its arm}$$

$$P = 2 \text{ lbs.},\quad W = 20 \text{ lbs.};$$

$$OA = \text{power-arm} = 10 \text{ ft.}$$

$$CB = \text{weight-arm} = x \text{ ft.}$$

By substitution,

$$2 \times 10 = 20 \times x$$

$$\therefore 20 = 20x$$

$$\therefore x = 1.$$

The weight-arm CB is, therefore, 1 ft. in length and the weight must be placed at the distance of one foot from the fulcrum.

2. In a lever of the second kind whose length is 10 inches, where must the weight be placed if P = 12 lbs. when W = 15 lbs.?

Ans. 8 in. from the fulcrum.

3. What force may be applied at one end of a lever 30 centimetres long to raise a weight of 30 kilogrammes hanging 6 centimetres from the fulcrum which is at the other end and what is the preserve on the fulcrum? *Ans.* 6 and 24 kilos.

4. A lever 30 ft. long rests above and upon a beam at one extremity; at the other end a force of 5 lbs. acts upwards. Where must a weight of 25 lbs. be placed to preserve equilibrium?

Ans. At 6 ft. from the fulcrum.

5. If the arm of a cork compresser be 12 inches, and a cork be placed at a distance of $1\frac{1}{2}$ inches from the fulcrum, find the pressure produced by a weight of 100 lbs. suspended from the handle. *Ans.* 800 lbs.

6. One extremity of a straight lever (without weight) whose length is 20 ft. rests upon a fulcrum ; at what distance from the fulcrum must a weight of 112 lbs. be placed, so that it may be supported by a force of 50 lbs. acting at the other extremity ? *Ans.* $8\frac{13}{14}$ ft.

7. What force must be applied at one end of a lever 12 ft. long to raise a weight of 100 lbs. hanging 5 ft. from the fulcrum which is at the other end ? State also what pressure is produced upon the fulcrum. *Ans.* $41\frac{2}{3}$ lbs.; $58\frac{1}{3}$ lbs.

8. A uniform lever 20 ft. long having one end fixed is in equilibrium with a force of $2\frac{1}{2}$ lbs. applied upwards at the other end ; what is the weight of the lever ? *Ans.* 5 lbs.

9. If the oar of a boat be 12 ft. long and the rowlock be 2 ft. from the handle, compare the pull of the oarsman with the resistance of the boat *Ans.* 5 : 6.

10. How many oarsmen will be required to row a boat whose resistance is 4 cwt, when each oar is 12 ft. long and the rowlock 3 ft. from the handle, supposing each of them pulls with a force of 112 lbs. *Ans* 3.

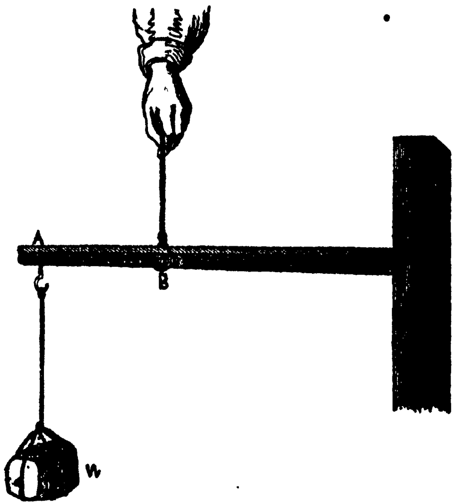
11. A wheel barrow weighing 20 lbs is 5 ft. long from its handles to the axle of its wheel ; find the force exerted by each of the workman's hands when the barrow is in use with a weight of 160 lbs. placed in it, the centre of gravity of the barrow and load being at a distance of $1\frac{1}{2}$ ft. from the axle ? *Ans* 27 lbs.

12. The length of a lever of the second kind is 16 inches, the weight arm = 4 inches, P = 30 lbs., and W = 90 lbs. Find the weight of the lever ? *Ans.* 15 lbs.

13. A bar 4 ft. long and weighing 2 lbs. is used as a lever of the second kind, the weight of the bar acting concurrently with W. What is P if W be 180 lbs. and W's arm be 8 ins.? *Ans.* 31 lbs.

14. What force must be applied at the end of a lever 20 inches long to raise a weight of 80 lbs. slung at a point 5 inches from the other end, if the weight of the lever is 4 lbs. and acts at its middle point ? *Ans* $9\frac{1}{2}$ lbs.

EXAMPLES OF LEVER OF THE THIRD KIND.



1. A straight rod 6ft. long, projects horizontally from a wall to which it is attached by a horizontal hinge acting without friction. A weight of 10 lbs. is attached to the extremity of the rod. What force or pull upwards must be exerted on the rod at 4 feet from the hinge, in order to retain the rod in its horizontal position?

The lever is of the third kind, and, for equilibrium, P × its arm = W × its arm.

$$\begin{aligned} BC &= \text{the power-arm} = 4 \text{ ft.} \\ AC &= \text{the weight-arm} = 6 \text{ ft.} \\ W &= 10 \text{ lbs.} \\ P &= x. \\ \therefore x \times 4 &= 10 \times 6 \\ 4x &= 60 \\ \therefore x &= 15 \end{aligned}$$

∴ the power is one of 15 lbs.

2. In a lever of the third order, the weight is 100 lbs. and rests 2 feet from the fulcrum. What power will be required if it acts at the centre of the lever? *Ans.* 200 lbs.

3. In turning a lathe, the foot of the operator acts at a distance of 2 ft. from the fulcrum, the longer arm is 5 ft. in length, the resistance is 20 lbs., what is the force? *Ans.* 50 lbs.

4. A lever with the fulcrum at one end has arms such that one is 3 ft. longer than the other; if the power is 10 times the weight, what is the length of the lever? *Ans.* $3\frac{1}{3}$ ft.

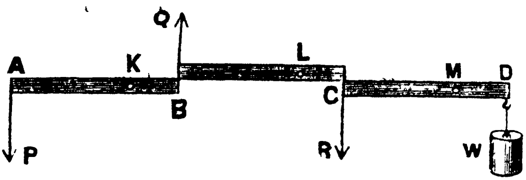
5. In a lever of the third kind, the power is $3\frac{1}{2}$ times the weight and the pressure on the fulcrum is 18 lbs. what is the weight ? *Ans.* $7\frac{1}{5}$ lbs.

6. In a lever with the power between the fulcrum and the weight the pressure on the fulcrum is 15 lbs. and the sum of the power and weight is 25 lbs. If the distance between the points of application of the power and weight be 6 in. find the length of the lever. *Ans.* 8 in.

7. In a lever, 9 ft. long with the fulcrum at one end and the weights at the other, the pressure on the fulcrum is 10 lbs. Find the power and the weight, when the power acts 3 ft. from the fulcrum. *Ans.* 15 lbs ; 5 lbs.

8. A straight heavy lever, 21 inches long, has its fulcrum at one end. A power of 12 lbs. acting vertically at a di stance of 7 in. from the fulcrum, supports a weight of 3 lbs. hung at the other end of the lever. If the weight be increased by 1 lb., what power acting at a distance of 5 in. from the fulcrum, will support it ? *Ans.* 21 lbs.

56. Combination of levers.—Several levers may be used in juxta-position either to increase the mechanical advantage, *i.e.*, to support a very large weight with a small power, or else to magnify small movements ; we are here concerned only with the former aspect of the problem.



Condition of Equilibrium.—Let AB, BC and CD be three levers in equilibrium moving round pivots K, L and M respectively.

If Q be the pressure at B between the two levers which are in contact there, and R the pressure at C between the two levers which are there in contact, then since the lever AKB is in equilibrium,

$$P \times AK = Q \times BK$$

$$\text{or } \frac{Q}{P} = \frac{AK}{BK},$$

and since the lever BLC is in equilibrium,

$$Q \times BL = R \times CL$$

$$\text{or } \frac{R}{Q} = \frac{BL}{CL},$$

and also since the lever CMD is in equilibrium,

$$R \times CM = W \times DM,$$

$$\text{or } \frac{W}{R} = \frac{CM}{DM};$$

therefore, $$\frac{Q}{P} \times \frac{R}{Q} \times \frac{W}{R} = \frac{AK}{BK} \times \frac{BL}{CL} \times \frac{CM}{DM},$$

$$\text{or } \frac{W}{P} = \frac{AK \times BL \times CM}{BK \times CL \times DM},$$

hence, $P \times AK.BL.CM = W \times BK.CL.DM$;

i. e., **the product of P into the several power-arms is equal to the product of W into the several weight-arms.**

EXAMPLES.

1. If there be a combination of levers of the first kind, with long arms of 14, 10 and 16 inches, respectively, and short arms of 3, 4 and 3 inches, respectively, what weight will be balanced by a power of 20 lbs.?

By the principle of compound levers—

P × the product of the several power-arms = W × the product of the several weight-arms;

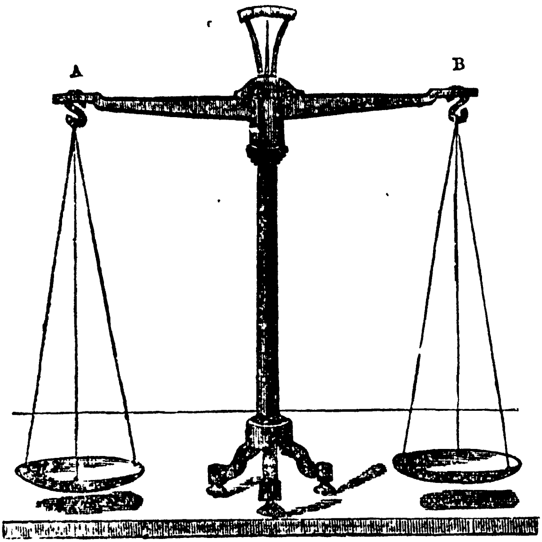
therefore, by substitution,

$$20 \times (14 \times 10 \times 16) = W \times (3 \times 4 \times 3)$$
$$44800 = 36\,W$$
$$W = 1244\tfrac{4}{9};$$

therefore, a weight of $1244\frac{4}{9}$ lbs. will be balanced by the 'power'.

2. Four levers, whose arms are severally 10 and 2, 8 and 3, 15 and 1, 12 and 2, are used in combination; what is the mechanical advantage? *Ans.* 1,200.

57. The common balance, its construction.—



The balance is an instrument for weighing bodies. It essentially consists of a lever of the first order, AB, which is called the **beam** with equal and similar arms ; it is supported at the middle of its length at a point called the **fulcrum**. From the two ends of the beam are suspended two **scale-pans** which always hang vertically, in one of these the object to be weighed, *i, e.*, the **load** is placed while the other contains the standard 'weights' or counterpoises with which it is to be compared.

It is important that the beam be freely moveable about its point of support; with this view the beam is furnished with a steel knife-edge, which is fixed transversely at its centre and projects out on each side, these projecting edges rest upon two small horizontal planes of agate or other hard material, one in front of the lever and the other

behind it; and when the beam is oscillating, it turns about these edges. At the extremities of the beam two other steel knife-edges are fixed facing upwards; these edges support flat planes from which the pans are suspended by wires or chains. The fulcrum-edge and the two edges at the extremities are all in one straight line. A pointer called the **tongue**, fixed at right angles to the beam, moves in front of a graduated arc; this shows the position the beam occupies or the angle through which it is displaced from the horizontal.

The instrument is so constructed that *equal weights* are indicated by the balance taking up the *horizontal* position.

58. Requisites of good balance.—In order that a balance should discharge its functions properly, it is necessary that it should be (i) **true** or just, (ii) **sensitive** and (iii) **stable.**

Def.—A balance is **true** or **just**, if its **beam** remain **horizontal** both when the pans are empty and when they are equally loaded.

Def.—A balance is said to be **sensitive**, when a **small difference** in the loads is indicated by a **large movement** of the beam.

Def.—A balance is said to possess **stability**, if after being displaced it **regains** quickly its position of equilibrium.

59. Conditions to be satisfied by a true balance.—In a good balance the points of support of the scale-pans and the fulcrum, *i. e.*, points A, B and C are in one straight line. Considered as a lever, the forces acting on it are the weights (S and S′) of the two scale-pans, the loads placed in the pans and the weight W of the

beam. The beam being horizontal, W acts vertically in the line passing through C, the weight of the beam, therefore, has no moment round the fulcrum and may be dismissed from further consideration.

Let the arms of the beam be a and a', then taking moments round the fulcrum $S.\,a = S'.\,a'$.

Suppose now that two equal loads P and P are placed, one in each pan, then again taking moments

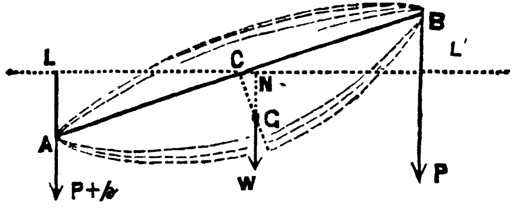
$$(P + S)\,a = (P + S')\,a',$$
$$\text{but } S.\,a = S'.\,a'$$
$$\therefore\ P.\,a = P.\,a',$$
$$\text{hence } a = a'.$$

Also since $S.\,a = S'.\,a'$ (*vide supra*), it is necessary that $S = S'$.

In other words if (i) *the two arms of the balance are of equal length*, and (ii) *the scales are of equal weight*, then with equal loads the balance remains horizontal and, by definition, *the balance is just.*

To test the justness experimentally, we have only to place a certain load in one pan and balance it by putting sand or any similar substance in the other ; if we now interchange the masses and if the beam still remains horizontal, the balance is just, because it satisfies the condition laid down by the definition.

60. Conditions which favour sensitiveness.—



We first assume that the balance is true. Let the centre of gravity of the balance be at G at the distance C G below the fulcrum C and let one weight be greater than the other by the amount p; then for the given difference p, the greater the deviation of the beam from the horizontal position, the greater is the sensitiveness of the balance. Eliminating equal weights P and P on the two sides, it is evident that round the point C, the moment of the excess weight p acting at A is opposed by the moment of W, the weight of the balance, acting at G and if the moment of W can be lessened or that of p increased, the deviation of the beam will increase and the sensitiveness of the balance will be greater.

Now the moment of W is equal to W × CN, this we can lessen (i) by lessening W, *i.e.*, by reducing the weight of the empty balance and also (ii) by shortening CN which can be done by bringing G nearer C. As p is a fixed quantity, the moment of p ($=p \times$ CL) can be increased only in one way, *viz.*, by increasing CL, which can be accomplished only by increasing the arms CA and CB of the balance. Thus the sensitiveness may be made considerable either—

(1) by reducing the weight of the beam and pans;

or (2) by bringing the centre of gravity of the beam nearer the fulcrum;

or (3) by increasing the length of the arms.

61. Condition requisite for stability. Circumstances favouring stability.—If after loading the pans equally, the balance is disturbed by the hand, then the only moment tending to restore horizontality will be

the moment of W, and for W to have a moment round C in the right direction, it is absolutely necessary to have G below C; hence the condition of stability is that *the centre of gravity of the balance must be always below the fulcrum.*

Now, the greater the moment of W (*i. e.*, W × CN), the greater will be the stability; therefore for great stability both W and CN must be large, *i.e.*, the *weight* of the balance must be *considerable* and the *centre of gravity* must be *well below* the fulcrum: these conditions, however, are detrimental to high sensitiveness. A compromise, however, may be effected and sufficient stability with fair sensitiveness secured by making the distance CG not very small, *i.e.*, by keeping the centre of gravity sufficiently low, and thus securing requisite stability and further by giving the balance long arms and thus increasing its sensitiv eness; but increasing the length of the arms will increase the weight of the beam and thus nullify the advantage sought to be gained; this is, in a great measure, prevented by making the beam in shape like an elongated lozenge, a form calculated to give rigidity with lightness.

To recapitulate :—

Conditions favouring sensitiveness are	Conditions favouring stability are	Compromise effected by having
(*a*) C. G. near the fulcrum, (*b*) small weight of balance, (*c*) long arms.	(*a*) C. G. low down, (*b*) large weight of balance.	(*a*) C. G. fairly low, (*b*) long but light arms.

62 False Balance. Double weighing.—A balance is said to be 'false,' when it does not indicate the true or correct weight of a body. In a false balance one arm is usually longer than the other.

We will now proceed to indicate how the true weight of a body can be ascertained by such a balance.

Let W be the *true weight* of a body, and a and b the respective lengths of the two unequal arms.

When the body is placed in the pan suspended from the a-arm, if a marked weight P is required to balance it, then as on the principle of the lever, for equilibrium

$$P \times \text{its arm} = W \times \text{its arm},$$
$$\therefore W \times a = P \times b\,;$$

similarly when it is suspended from the arm b, if a marked weight P′ is required to balance it,

$$\text{then } W \times b = P' \times a\,;$$

therefore, combining the two equations, we get

$$W^2.\ a.b = P.\ P'.\ a.\ b,$$
$$\text{and } W^2 = P.\ P'\,;$$
$$\therefore W^2 = \sqrt{PP'}.$$

Hence, the **true weight is a mean proportional between the two false weights.**

In practice, therefore, if we successively weigh the object in both the pans and extract the square-root of the product of the counterpoises in the two instances, we ascertain the true weight of the object. This is the method of Gauss.

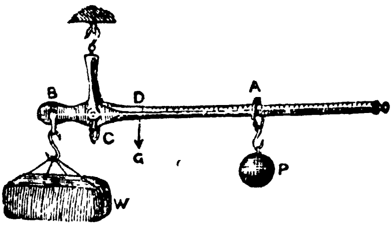
We can also ascertain by the above process the extent of the inequality in the two arms.

$$\text{For since } W \times a = P \times b$$
$$\text{and } P' \times a = W \times b,$$
$$\therefore P'.\ W.\ a^2 = P.\ W.\ b^2,$$
$$\text{and } P'.\ a^2 = P.\ b^2\,;$$
$$\therefore \frac{a^2}{b^2} = \frac{P}{P'} \quad \text{and } a : b :: \sqrt{P} : \sqrt{P'}\,;$$

i.e., the two arms are proportional to the square-roots of the counterpoises required to balance the body.

Borda's Method of obtaining the true weight of a substance with a false balance is to place the body in one scale-pan and to pour sand or shots into the other till the beam becomes horizontal and equilibrium is established; then remove the body from the scale-pan and replace it with *marked weights* till equilibrium is again established. It is clear that *the marked weights used represent the weight of the body, because they produce exactly the same effect as the body itself.*

63. The Roman Steel-yard.—The common or Roman steel-yard is a **balance** with **arms unequal in length and weight..** It is suspended from a point C which is the fulcrum.



At the point B in the **shorter arm** is attached a hook from which is suspended **the substance** to be weighed, while a ring carrying a **constant counterpoise P, slides along the divided arm** on the right.

As the 'yard is of unequal thickness the centre of gravity is not in the middle of the rod, but nearer the thicker end, as at D, where the weight of the yard, G, acts. the steel-yard is in equilibrium when, with a certain load hanging from B and the counterpoise P from a point such as A, the 'yard' occupies the horizontal position.

Method of Graduation.—Taking moments round C,

$$W \times CB = P \times CA + G \times CD,$$

$$\text{and } W \times CB - P \times CA - G \times CD = 0 \quad \text{(i)}$$

For any other load **W'**, **P** will have to be shifted to a new position **A** either to the right or left of **A** as W' is greater or less than **W**, and

$$W' \times CB - P \times CA' - G \times CD = 0 \quad \text{(ii)}$$

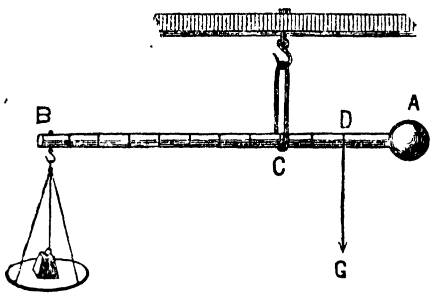
Let W' less than W, then subtracting (ii) from (i) we get

$$(W - W') \, CB - P \, (CA - CA') = 0$$

$$\text{or } (W - W') \, CB = P . AA'.$$

In the above equation, it will be observed that as CB and P are constant and invariable, to maintain equality, (W – W') must vary as AA', that is, *the change in the weight of the load must be proportional the change in the length of the power-arm CA*. Hence we may graduate thus:—hang any known weight, say 5 lbs., at B and mark 5 at A, the position of equalibrium of P; next hang 4 lbs. at B and mark 4 at A', the new position of, equilibrium of P; set off distances, each equal to the distance between 4 and 5 to the left and right respectively of 4 and 5, then for every difference of a pound, P will have to be shifted through one division, and the position of P on the long arm will give the weight of the substance in pounds.

64. The Danish Steel-yard—In this from of the steel-yard one end A is weighted, this acts as the counter-poise; while the fulcrum which is usually in the form of a ring is move-able. The empty yard is horizontal, *i.e.*, in equilibrium when the plane of the supporting ring or fulcrum is in the same vertical line as the centre of gravity of the yard; such point is D. If now a series of weights be successively placed in the pan to restore equilibrium, the fulcrum will have to be moved further and further away from the weighted end A. If these points are once for all marked and numbered the steel-yard is graduated permanently.



EXAMPLES.

False balance.

1. A body placed in one scale of a false balance weighs $4\frac{1}{2}$ lbs., while in the other scale it weighs $5\frac{5}{9}$ lbs. Find its true weight.

Let W be the true weight of the body and let a and b be the unknown arms of the balance.

When the body is placed in one pan, it is balanced by a weight of $4\frac{1}{2}$ lbs., and when placed in the other, by a weight of $5\frac{5}{9}$ lbs.

Then since the weights W and $4\frac{1}{2}$ are in equilibrium,

$$W \times a = 4\tfrac{1}{2} \times b.$$

Similarly $$W \times b = 5\tfrac{5}{9} \times a.$$

Compounding $$W^2 \times a.b = 4\tfrac{1}{2} \times 5\tfrac{5}{9} \times a\,b,$$

$$\therefore W^2 = 25$$

$$\text{and } W = 5;$$

hence, the true weight of the body is 5 lbs.

2. A body when put into one scale-pan weighs 8 seers, and when put into the opposite one weighs $4\frac{1}{2}$ seers. Find the true weight of the body

Ans. 6 seers.

3. In testing the correctness of a balance a certain quantity of rice was found to weigh $6\frac{1}{4}$ lbs. when put in one pan, while it only weighed 4 lbs. when put in the other; required the true weight of the rice.

Ans. 5 lbs.

4. A body, the weight of which is 12 lbs., when placed in one scale of a false balance appears to weigh 9 lbs,; find its weight when placed in the other scale.

By article 62' $W^2 = P.\,P'$, where W is the true weight of a body and P P', the apparent weights.

$$\therefore 12^2 = 9 \times P'$$

$$\therefore \quad P' = \frac{12 \times 12}{9} = 16$$

Ans. 16 lbs.

5. A body, the weight of which is 42 lbs., when placed in one scale of a false balance appears to weigh 98 lbs.; find its weight when placed in the other scale. *Ans.* 18 lbs.

6. A pound weight appears to weigh 14 ozs. when placed in one scale of a balance with unequal arms; find its apparent weight when placed in the other scale. *Ans.* 18. 3 ozs.

7. A piece of lead placed in one pan A of a balance is counterpoised by 8 lbs. in the other pan B. When the same piece of lead is placed in the pan B it required $4\frac{1}{2}$ lbs. in A to balance it. Find the ratio between the arms.

$$\text{True weight} = \sqrt{8 \times \tfrac{9}{2}}$$
$$= 6 \text{ lbs.}$$

Let a and b represent the two arms

$$\text{then } 6a = 8b$$
$$\therefore\ 3a = 4b$$
$$\therefore\ a : b :: 4 : 3$$

8. A substance is weighed from both arms of an unequal balance and its apparent weights are 9 lbs. and 4 lbs.; find the ratio between the arms.

Ans. $a : b :: 2 : 3$.

9. A substance is weighed from both arms of an unequal balance, and its apparent weights are 9 and 16 kilo-grammes. Find the ratio between the arms. *Ans.* 3 : 4.

10. In a false balance a weight is measured in one scale by 9 lbs. and in the other by 49 lbs. If the longer arm is 14 in. find the length of the beam. *Ans.* 20 in.

11. In a false balance the apparent weights of a body are $42\frac{1}{4}$ lbs. and 49 lbs., and the whole length of the beam is $2\frac{1}{4}$ ft.; find the length of each arm. *Ans.* 13 in.; 14 in.

12. The ratio between the arms of a false balance is 6 : 7. A body suspended from the extremity of the longer arm is balanced by 14 lbs. in the other pan. Find the true weight of the body. *Ans.* 12 lbs.

13. The arms of a balance are in the ratio of 32 to 33., and the pan in which the weights are placed is suspended from the longer arm; what is the real weight of a body that apparently weighs one pound.

Ans. $16\frac{1}{2}$ oz.

14. The arms of a false balance are one decimetre and 1·05 decimetre in length; what will be the apparent weight of 10 kilogrammes of goods weighed with this balance, when suspended from the longer arm?
Ans. 105 kilograme.

15. The arms of a false balance are to each other as 7 to 8 and the weight is put into one scale as often as into the other; what will be the gain or loss per cwt. to the seller?

Let P be the counterpoise and W the actual or *true* weight of the substance weighed out. Then as on the principle of the lever, P $\times$ its arm = W $\times$ its arm.

therefore, at one time $P \times 7 = W \times 8$

$$\text{and } W = \tfrac{7}{8} P;$$

and at the other time $P \times 8 = W \times 7$

$$\text{and } W = \tfrac{8}{7} P.$$

The total true weight thus dealt out in the two weighings is $(\frac{7}{8}+\frac{8}{7})P = 2\frac{1}{56}P$, while the counterpoise only indicates 2 P; therefore at every two alternate weighings, nominally equal to 2 P, the seller loses in price equivalent to that of $\frac{1}{56}$ P. On every 2 cwts. he sells, his loss is hence equal to the price of $\frac{1}{56}$ of 1 cwt, *i.e.*, two lbs, and consequently the loss per cwt. is equal to the price of one pound.

16. A person purchased 2 *shers* of sugar, and suspecting some fraud in the balance, he had one *sher* nominally weighed in one of the scales, and the second *sher* in the other scale; how much did he gain or lose by the bargain, supposing the lengths of the arms to be at 6 : 5? *Ans.* He gained $\frac{1}{30}$ of a *sher.*

17. A tradesman's balance has arms whose lengths are 11 ins. and 12 ins. respectively, and it rests horizontally when the scales are empty. If he sell to each of two customers a pound of tea at Re. 1 as. 8 per pound, putting his weights into different scales for each transaction, find whether he gains or loses owing to the incorrectness of his balance, and how much *Ans.* $2\frac{2}{11}$ pies.

18. A fraudulent grocer sells sugar at Rs. 28 per cwt., and uses a balance, the arms of which are respectively 18 and 19 inches long. Calculate the profit per pound accruing to the grocer on account of the fraud.

Evidently for obtaining illicit profit the sugar is weighed out in the pan suspended from the longer arm.

Now, on the principle of fhe lever,

$$P \times \text{its arm} = W \times \text{arm},$$
$$\therefore P \times 18 = W \times 19$$

where P is the counterpoise and W the true weight of the sugar.

$$\therefore \frac{P}{W} = \frac{19}{18}.$$

Therefore for every 18 lbs. of sugar that the grocer actually deals out the counterpoise indicates 19 lbs.

Now, as the price of the sugar is Rs. 28 per cwt. or 4 annas per lb., and as the grocer charges the price of 19 lbs. for every 18 lbs. that he actually sells, therefore, by his fraudulent practice he gains extra 4 annas on every 18 lbs. of sugar, and for every lb. of sugar the illicit profit is $\frac{4}{18}$ of an anna or $\frac{3}{9}$ of a pice.

19. If a balance be false with arms in the ratio of 7:8, find how much, per pound a customer really pays for tea which is sold to him from the longer arm at 7 annas per pound. *Ans.* 8 annas per pouud.

BALANCE WITH SCALE-PANS OF UNEQUAL WEIGHTS.

1. In a balance with equal arms and unequal scale pans, show that the true weight is half the sum of two apparent weights when weighed successively in the two scale pans.

Let W be the true weight of the body, P and P′ its two apparent weights, S and S′ the weights of the two scale-pans and a the length of each arm; then taking moments about the fulcrum in the two cases.

$$(W+S)a = (P+S')a \quad . \quad . \quad . \quad \text{(i)}$$
$$(P'+S)a = (W+S')a \quad . \quad . \quad . \quad \text{(ii)}$$

Subtracting (ii) from (i)

$$W-P' = P-W,$$
$$2W = P+P',$$
$$\text{and } W = \tfrac{1}{2}(P+P').$$

Hence the true weight is half the sum of the apparent weights.

2. The arms of a false balance are equal. A body weighs 5 lbs when placed in one pan and 7 lbs. when placed in the other. Find the true weight of the body.

$$\text{Here } W = \frac{P+P'}{2}$$
$$= \frac{5+7}{2} = 6$$

Ans. 6 lbs.

3. If a substance weighs apparantly 8 lbs. when placed in one scale of a false balance, and 6 lbs. when placed in the other, find how much one scale-pan is heavier than the other, if the arms are of equal length.

Ans. 1 lb.

4. The arms of a balance are of equal length, but it is found that a weight placed in one scale weighs apparently 9 lbs., and when placed in the other 8 lbs.; find the real weight of a body which weighs, when placed in the first scale-pan, 10 lbs. 8 oz. *Ans.* 10 lbs.

5. In a false balance 12 lbs. 10 oz. placed in one pan balance 13 lbs. in the other, find the defect in the balance.

Ans. The scale pans differ in weight by 6 ozs.

6. If the beam of an empty balance be not horizontal show that if want of horizontality arise from inequality in the weights of the scale-pans, the balance may be corrected by putting a weight into the lighter pan; but if it arise from a difference in length of the arms, the balance cannot be so corrected.

A balance is correct or true if the beam remains at rest in the horizontal position when the contents of the scales are interchanged. (*vide* article 56.)

In a balance of equal arms of length a, on the principle of the lever, for equilibrium, P $\times$ its arm must be equal to W $\times$ its arm; but if W, the weight of one scale-pan, be less than P, the weight of the other, and consequently W–arm titled upwards, then by adding a weight Q to W we can make $P = W + Q$,

$$\text{and } P \times a = (W + Q) \times a.$$

The balance will then not only be in equilibrium and horizontal but also correct, for if we next put two equal weights w in each of the pans,

$$(P + w) \times a = (W + Q + w) \times a,$$

and on interchanging the two weights w, the condition of equality will not be altered.

But if the arms have unequal lengths a and b, then initially horizontality may be obtained by making

$$P \times a = (W + Q) \times b,$$

but on putting equal weights w in each of the pans,

$(P + w) \times a$ will not be equal to $(W + Q + w) \times b$, because $w \times a$ is not equal to $w \times b$, and also on interchanging the weights the condition of equality will not be secured. The balance, therefore, will not be corrected or made true in the second case by adding a weight to the lighter pan.

QUESTIONS.

1. What is a *lever?* What are its *arms* and its *fulcrum?*

2. When is a lever said to be *straight* and when *bent?* When is a lever said to be *light?*

3. In what classes are levers divided and on what principle are the divisions based?

4. Describe the lever of the first kind. Give some examples of the same stating in each case how the 'power' and the 'weight' act.

5. Show that the oar is lever of the second kind and compare it with a crowbar employed to lift a weight.

6. State the positions of the 'power,' the 'weight' and the fulcrum in levers of the second and the third kinds. Also give examples of each kind.

7. Enunciate the *principle of the lever* and prove it for each of the hree 'kinds', (*a*) by taking moments and (*b*) on the principle of energy.

8. Indicate how far mechanical advantage is gained or lost in the use of the three kinds of levers.

9. What is a *compound lever.* What are its functions? What is the condition of equlibrium?

10. Describe the *Common balance.*

11. What are the three essential conditions or requisites that a good balance must satisfy? How is each of them secured irrespective of the other two?

12. Which of these conditions is it impossible to secure together to the highest degree and why? How is this remedied in practice?

13. What is a *false* balance? What is the most common defect of construction in a false balance?

14. Explain Gauss's method of *double weighing* and show that the effect of any inequality in the arms of the balance is thereby eliminated.

15. Give *Borda's method* of weighing.

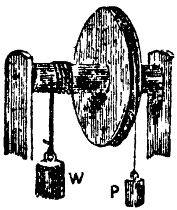
16. What is a *steel-yard?* Distinguish between the 'Roman' form of it and the 'Danish'.

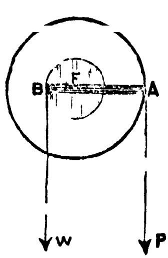
17. What kinds of levers do the following illustrate? (1) The handle attached to the rudder of a ship. (2) A poker when used for poking fire. (3) A cork-compressor?

CHAPTER III.

The Wheel and Axle.

65. The Simple Wheel and Axle.—A lever can no be used to raise weights *far*, but an easy modification, securing **continuous action,** is to make the **fulcrum into a pivot** and to **apply P and W at the circumferences of two unequal circles** or wheels with a common centre; thus we get what is called the *wheel and axle.* By this means the action of a lever is continued as long as we may please, the weight rising all the time. The **weight hangs on** a cord round the circumference of the smaller circle called the **axle**, and the **power is applied** to another cord wound in the **opposite direction round** the larger circle called the **wheel.**





Condition of Equilibrium.—Viewed in section, we have the power P acting at the end of the radius F A of the large wheel and the weight W at the end of the radius FB of the small wheel or axle. Then it is evident that AFB is a **lever of the first kind** and the condition of equilibrium is

(*i*) *by taking moments* round F,

$$P \times FA = W \times FB,$$

$$\therefore \frac{W}{P} = \frac{FA}{FB} = \frac{\text{radius of the wheel}}{\text{radius of the axle}}.$$

Or, (*ii*) *on the 'principle of energy,'* as at every revolution of the wheel, the axle also makes one revolution,

> therefore, when P descends through a distance equal to that of the circumference of the wheel, W ascends through a distance equal to that of the circumference of the axle ;

and the *equation of work* becomes

P $\times$ circumference of the wheel = W $\times$ circumference of the axle,

$$\text{or } \frac{P}{W} = \frac{\text{cir. of the axle}}{\text{cir. of the wheel}};$$

and as the circumference of circles are proportional to their radii,

$$\therefore \frac{P}{W} = \frac{\text{radius of the axle}}{\text{radius of the wheel}},$$

and $\frac{W}{P}$ or **mechanical advantage** $= \frac{\text{radius of the wheel}}{\text{radius of the axle}}$.

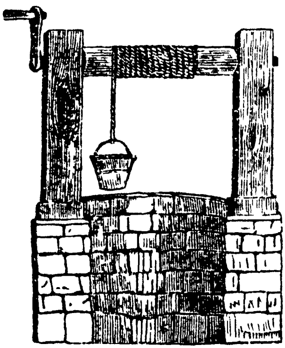
In the above investigation, we have not taken the **thicknesses of the ropes** into consideration. However, if compared with the radii of the wheel and the axle, they are too great to be neglected, then we must suppose that the forces are acting at the centres of the two ropes, and in the above formula we must add to the radii half the thicknesses of the ropes. The formula then takes the from

$$\frac{W}{P} = \frac{\text{sum of the radii of the wheel and its rope}}{\text{sum of the radii of the axle and its rope}}.$$

66. Limit to Mechanical Advantage.—The mechanical advantage depends upon the value of $\frac{W}{P}$, which depends upon the ratio $\frac{\text{radius of wheel}}{\text{radius of axle}}$. (*Vide supra.*) Therefore, by making the wheel large and also by making the axle

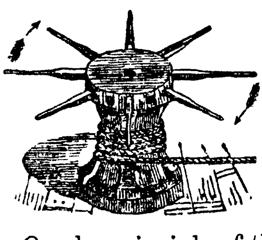
thin, we can greatly increase the advantage ; but we cannot do so indefinitely, because a limit is reached when the wheel becomes unwieldly in size or the axle too thin to support the 'weight.' Hence the size of the 'wheel' and the strength of the material of which the 'axle' is made limit the extent of the mechanical advantage that can be derived.

67. The Windlass and the Capstan.—The **windlass** is a modification of the wheel and axle in which the **power** is **applied** not at the circumference of a 'wheel,' but at the extremity of a **handle**, which has a length greater than the radius of the axle, which is called the 'barrel.' This contrivance is often used for raising water from wells.



On the principle of the 'wheel and axle,' P × length of the handle = W × radius of the barrel, $\therefore \frac{W}{P} = \frac{\text{length of the handle}}{\text{radius of the barrel}}$.

The **capstan** is another modification of the 'wheel and axle.' In it, the **axle** is **vertical** and there are **several handles** instead of one to which 'power' is applied simultaneously. This machine is used on boardship for raising the anchor.



On the principle of the 'wheel and axle,' P × its handle + P′ × its handle + &c. = W × radius of the axle.

If all the handles are of equal length and the values of P the same, then in a capstan of n handles—

$$n\,(P \times \text{length of the handle}) = W \times \text{radius of the axle};$$

$$\therefore\ \frac{W}{nP} = \frac{\text{length of the handle}}{\text{radius of the axle}}.$$

The **windlass** is also sometimes fitted **with several handles**, in which case the condition of equilibrium is the same as that for the capstan.

EXAMPLES.

1. Given a 'windlass' having an axle 1 ft. in diameter and an arm $2\frac{1}{2}$ ft. in length, what amount of force would you apply to the handle in order just to support a weight of 10 cwts. 2 qrs. 24 lbs.?

The condition of equilibrium is—

$$\frac{W}{P} = \frac{\text{length of the handle}}{\text{radius of the axle.}}$$

10 cwt. 2 qrs. 24 lbs. = 1,200 lbs.

By substitution, $\frac{1200}{P} = \frac{2\cdot5}{0\cdot5}$

$\therefore\ 2\cdot5\,P = 600$ and $P = 240$ lbs.

2. What is the mechanical advantage of a wheel and axle in which the diameter or the axle is 5 centimetres and the radius of the wheel 2 decimetres? *Ans.* 8.

3. A wheel and axle have circumferences of 4 ft. 8. in. and of 10 ins. respectively. Find the power which will balance a weight of 4 cwts. *Ans.* 80 lbs.

4. A wheel and axle have radii respectively 3 ft. 4 in. and 4 in.; find the power which will support a weight of 2 cwts. *Ans.* $\frac{1}{5}$ cwt.

5. In the wheel and axle, if the radius of the wheel be 12 ft. and that of the axle half a foot, what power will be required to sustain a weight of 216 lbs.? *Ans.* 9 lbs.

6. What force will be required to work the handle of a windlass, the resistance being 1,200 lbs., the radius of the axle 6 ins. and that of the handle 2 ft. 6 ins.? *Ans.* 240 lbs.

7. If the handle of a windlass has a radius equal to four times the radius of the cylinder round which the rope is wound, determine the force which must be applied to the handle to raise a load of 200 lbs. *Ans.* 50 lbs.

8. If the handle or a windlass have a radius equal to five times the radius of the cylinder round which the rope is wound, determine the force which must be applied to the handle to raise a load of 1,000 kilogrammes. *Ans.* 200 kilos.

9. If a windlass be worked by two handles, each 3 ft. long, and if the diameter of the barrel upon which the rope coils be 6 ft. 4 ins., what force will be required upon each handle to support a weight of 1,000 lbs.? *Ans.* $527\frac{7}{9}$ lbs.

10. A windlass has a 'barrel' 1 ft. in diameter, and is worked by a handle 27 ins. distant from the axis of the 'barrel'; what power must be applied to support a bucket of water weighing 100 lbs.? *Ans.* $22\frac{2}{9}$ lbs.

11. A wheel and axle is used to raise a bucket from a well. The radius of the wheel is 15 in.; and while it makes 7 revolutions the bucket, which weighs 30 lbs., rises $5\frac{1}{2}$ feet. Find what is the smallest force that can be employed to turn the wheel. *Ans.* Slightly greater than 3 lbs.

12. The handle of a windlass has a radius equal to four times the radius of the axle. The axle carries two buckets of equal weight on opposite sides; with what force must the handle be turned to raise 5 gallons of water. A gallon of water weighs 10 lbs. *Ans.* $12\frac{1}{2}$ lbs.

13. Four blue-jackets raise the anchor of a ship by means of a capstan. The capstan is 3 ft. in diameter, and has 4 bars, each bar being 12 ft. long; the men press with a force of 100 lbs. at the extreme end of each bar, what is the maximum weight of the anchor?

$$\frac{W}{P} = \frac{\text{Radius of wheel}}{\text{Radius of axle}};$$

$$\frac{W}{4 \times 100} = \frac{12}{\frac{3}{2}}$$

$$\therefore W = 3200 \text{ lbs. } \textit{Ans.}$$

14. In a wheel and axle, the radius of the wheel is 30 in. and that of the axle $3\frac{3}{4}$ in.; if the power applied be 60 lbs., what must be the weight? *Ans.* 480 lbs.

15. Find what weight suspended from the axle can be supported by 5 lbs. suspended from the wheel, if the radius of the axle is 18 inches and the radius of the wheel is 4 ft. 6 inches.* *Ans.* 15 lbs.

16. The radius of the wheel being three times that of the axle, and the string on the wheel being only strong enough to support a tension equivalent to 30 lbs., find the greatest weight which can be lifted *Ans.* 90 lbs.

17. If the radius of the wheel be n times as great as that of the axle and the maximum tension of the string on the wheel be t, find the greatest weight that can be raised. *Ans.* $W = nt$.

18. If a man exert a pressure of 60 lbs. upon the handle of a windlass what weight will he raise when the length of the handle is 2 ft. and the radius of the axle 3 ins.? *Ans.* 480 lbs.

19. In the wheel and axle the radius of the wheel is 5 times the radius of the axle; what weight can be supported by a power of 10 lbs.? *Ans.* 50 lbs.

20. If the length of the arms of the capstan be ten times the radius of the barrel, and if six men each exert a pressure of 50 lbs. upon the extremities of the arms, what will be the tension produced on the rope? *Ans.* 3000 lbs.

21. A capstan is worked by 6 men at the end of levers 10 ft. long. They each push with a force of 50 lbs. and the radius of the barrel is 1 ft. 6 ins.; what strain is exerted upon the hawser? *Ans.* 2000 lbs.

22. The radius of the axle of a capstan is 3 feet and 5 men push each with a force of 100 lbs. on spokes 6 ft. long; what tension will they be able to exertion the rope which leaves the axle? *Ans.* 1000 lbs.

23. A man whose weight is 140 lbs. is just able to support a weight that hangs over an axle of 6 in. radius, by hanging to the rope that passes over the corresponding wheel, the diameter of which is 4 ft.: find the weight supported. *Ans.* 560 lbs.

24. If the length of the arms of the capstan be 12 times the radius of the cylinder, and if four men each exert a pressure of 50 kilogrammes upon the arms, what will be the tension produced on the rope? *Ans.* 2,400 kilos.

25. A capstan is worked by a man pushing at the end of a pole. He exerts a force of 50 lbs., and walks 10 ft. round for every 2 ft. of rope pulled in. What is the resistance overcome? *Ans.* 250 lbs.

26. A weight of 10 lbs. is supported by a certain power on a wheel and axle, the radii being 30 and 12 ins. respectively: if the radii were each shortened by 6 ins., find what weight would be supported by the same power. *Ans.* 16 lbs.

27. Ten sailors, exerting a force of 150 lbs. each, can just lift a ship's anchor by means of a 'windlass,' when the radius of the axle is 20 inches and the length of the lever 15 feet: find the resistance of the anchor in tons.

The condition of equilibrium is $\dfrac{W}{nP} = \dfrac{\text{length of the handle}}{\text{radius of the axle.}}$

By substitution, $\dfrac{W}{10 \times 150} = \dfrac{180}{20}$

$$\therefore 20\ W = 270{,}000$$

$$W = 13{,}500 \text{ lbs.} = 6\tfrac{3}{112} \text{ tons.}$$

28. The radius of a wheel is 4 ft., and the radius of the axle is 4 in.; a weight of 6 lbs. is suspended from the wheel and one of 8 lbs. from the axle on the *same side.* What weight must be suspended from the wheel to keep them in equilibrium? *Ans.* $6\frac{2}{3}$ lbs.

29. A weight of 17 lbs. just balances another of 79 lbs. on a wheel and axle; what will be the radius of the axle if that of the wheel be 17 inches?

The condition of equilibrium is $\dfrac{W}{P} = \dfrac{\text{length of the handle}}{\text{radius of the axle.}}$

By substitution, therefore, $\dfrac{79}{17} = \dfrac{17}{\text{radius of the axle}}$,

and the radius of the axle $= \dfrac{17 \times 17}{79}$ inches;

$$= 3\tfrac{52}{79} \text{ inches.}$$

30. What is the diameter of a wheel if a power of 3 ozs. is just able to move a weight 12 ozs. that hangs from the axle, the radius of the axle being 2 ins.? *Ans.* 1 ft. 4 ins.

31. If a weight of 20 kilogrammes be supported on a wheel and axle by a force of 4 kilogrammes, and the radius of the axle is 5 centimetres, find the radius of the wheel. *Ans.* 25 cm.

32. Find the radius of the wheel to enable a power of 3 lbs. to support a weight of 33 lbs., the radius of the axle being 4 ins. *Ans.* 44 ins.

33. In the simple wheel and axle the radius of the axle is two inches, what must be the radius of the wheel if P = 10 when W = 160?

Ans. 2ft. 8 ins.

34. The handle of a windlass is 2 ft. long; what must be the radius of the axle so that a man exerting a pressure of 50 lbs. upon the handle may raise a bucket of water weighing 150 lbs.? *Ans.* 8 ins.

35. The handle of a windlass is 15 ins. distant from the axis; what must be the diameter of the barrel when a man exerting a force of 70 lbs. on the handle, supports a weight of 350 lbs.? *Ans.* 6 ins.

36. If the difference between the diameter of a wheel and the diameter of its axle be six times the radius of the axle, find the greatest weight that can be sustained by a force of 60 kilogrammes. *Ans.* 240 kilos.

37. The difference of the diameters of a wheel and axle is 2 ft. 6 ins. and the weight is equal to six times the power: find the radii of the wheel and the axle. *Ans.* $r = 3$ ins., $R = 18$ ins.

38. A weight of 3 cwts. is raised from the bottom of a mine by means of a force of 112 lbs.; what relation must the wheel and axle bear to each other to do this? *Ans.* 3 : 1.

39. A man whose weight is 12 stones has to balance by his weight 15 cwts.; show how to construct a wheel and axle which will enable him to do this. *Ans.* The radius of the wheel must be ten times the radius of the axle.

40. Draw (to scale) a simple wheel and axle, in which 1 lb. supports $7\frac{1}{2}$ lbs.

41. If the radii of the wheel and axle be respectively 2 ft. and 3 in. and if the thicknesses of the ropes be one inch, find what weight would be supported by a power of 8 lbs.?

The condition of equilibrium is—

$$\frac{W}{P} = \frac{\text{sum of the radii of wheel and its rope}}{\text{sum of the radii of axle and its rope}}.$$

Therefore, by substitution,

$$\frac{W}{8} = \frac{12+\frac{1}{2}}{1\frac{1}{2}+\frac{1}{2}} = \frac{12\frac{1}{2}}{2} = \frac{25}{4}; \therefore W = 50 \text{ lbs.}$$

42. A weight of one ton or 2,240 lbs. is sustained by a rope 2 ins. in diameter, going round an axle 4 inches in diameter; what weight must be suspended at the circumference of the wheel by a rope of the same thickness to obtain equilibrium, the radius of the wheel being 6 ft.? *Ans.* $92\frac{4}{73}$ lbs.

43. The radius of the wheel being 6 ft. and that of the axle 3 ins., find the weight supported by a power of 120 lbs., supposing the thickness of the rope coiled round the axle to be 1 in. *Ans.* $1251\frac{3}{7}$ lbs.

44. In the wheel and axle, if the radius of the wheel be 3 ft. and that of the axle 1 ft., a man is able to support a weight of 108 lbs. by pulling

the rope passing round the wheel vertically downwards. Find the pressure on the sockets. *Ans.* 144 lbs.

45. In the wheel and axle if the power acts upwards and on the same side of the axis as the weight, what is the pressure on the fixed supports?

Ans. W—P.

46. If the axle be uniformly square instead of round, compare the greatest and least weights which the same power can support in the Wheel and axel. *Ans.* $\sqrt{2} : 1$.

47. Is the mechanical advantage of a wheel and axle increased or diminished by lessening the radii of wheel and axle by the same amount?

Ans. Increased.

48. In what direction must the power act in order that the pressure on the axle may be the least possible, and the greatest possible?

Ans. (1) Vertically upwards; (2) vertically downwards.

49. The radius of the wheel is 11 times that of the axle, and when the weight is raised through a certain height it is found that the power has moved over 5 ft. more than the weight. Find the height through which the weight was moved. *Ans.* 6 in.

50 In the wheel and axle the radius of the wheel is 12 in. and that of the axle 5 in. and two weights of 10 and 5 lbs. respectively are suspended from their circumferences. Supposing the weight which tends to descend is supported by a small table, what will be the pressure on the table and the total pressure on the fixed supports on which the ends of the axis of the machine rest? *Ans.* $7\frac{11}{12}$; $7\frac{1}{12}$ lbs.

Taking the figure of § 65:—the force P = 10 must be supported on the table, for the downward moment of P about F is 120, and this is greater than the moment of W = 5 about F, which is 25.

If therefore R be the pressure on the table, then the three forces, P, W and R are in equilibrium, and

taking the moments about F,

$$P \times FA - R \times FA = W \times FB$$

$$10 \times 12 - R \times 12 = 5 \times 5$$

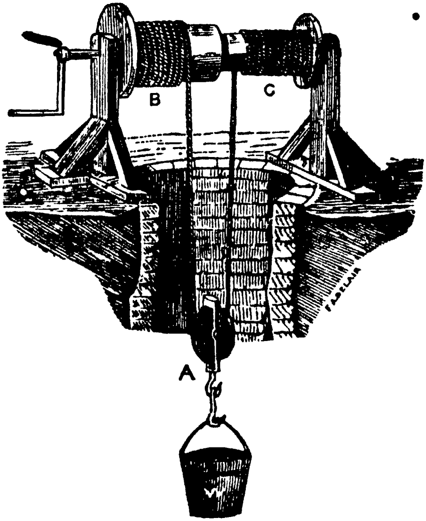
$$\therefore\ R = 7\tfrac{11}{12} \text{ lbs.}$$

The pressure on the supports of the axis is

$$= P - R + W = 10 - 7\tfrac{11}{12} + 5 = 7\tfrac{1}{12} \text{ lbs.}$$

Ans. $7\frac{11}{12}$ lbs. and $7\frac{1}{12}$ lbs. respectively.

68. The Chinese Windlass or Compound Wheel and Axle.



—In the ordinary windlass, **mechanical advantage** is measured by $\frac{\text{length of handle}}{\text{radius of axle.}}$ The **limit is reached** when the **length** of the **winch-handle cannot be** further conveniently **increased** and the **radius** of the **axle** further **reduced** without danger of breaking. *With the same size of the winch-handle*, **in order to obtain a greater mechanical advantage** than what is possible with the simple windlass, the **Chinese windlass or the compound wheel and axle** is **employed.** It is so called because it is extensively used in China for drawing out water from wells.

One-half, C, of the axle or barrel of an ordinary windlass is made with a smaller circular section than the other half, B. The two ends of a rope are attached to the parts B and C, and are coiled round the cylinder in opposite directions, in the way shown in the figure ; in the loop of the rope a moveable pulley with a hook is suspended. Then by turning the handles, as the rope is wound upon the cylinder B, it is at the same time unwound from the

cylinder C; owing, however, to the fact that the cylinder B is larger than C, the rope is wound up faster on B than it is unwound on C, and therefore the pulley A with its weight ascends.

Its **mechanical advantage.**—*On the 'principle of energy,'* which states that $P \times d = W \times d'$, in the compound wheel and axle, the *equation of work* becomes—

P × the circumference of the circle described by the handle = W × the space through which W is raised.

Now at each revolution as more of the rope is wound up than is let out, it is shortened by the difference between the circumferences of the two axles, and the weight is raised through half that difference; therefore the **space through which W is raised = one-half the difference between the circumferences of the two axles.**

By substitution, therefore,

P × the circumference of the circle described by the handle

$$= W \times \frac{\text{difference between the circumferences of the two axles.}}{2}$$

But as the circumferences of circles are proportional to their radii,

$\therefore$ P × length of handle

$$= W \times \frac{\text{difference between the radii of the two axles.}}{2}$$

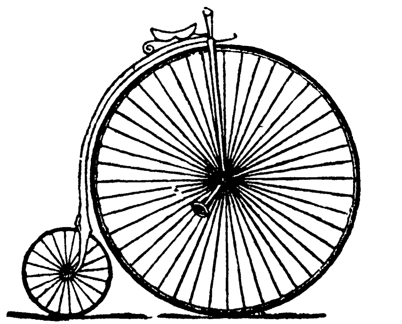
$$\text{or} \frac{W}{P} = 2 \times \frac{\text{length of the handle}}{\text{difference between the radii of the two axles.}}$$

Thus it will be seen that the mechanical advantage, which is equal to $\frac{2 \times \text{length of the Handle}}{\text{diff. between the radii}}$, is *unlimited* in this case, for without unduly increasing the length of the handle, the difference between the radii, of the axle can be made as small as possible without sacrificing the rigidity of the axle.

As the pull of the weight is equally borne by both parts of the rope, the **tension** on each side of the rope $= \frac{1}{2}$ W.

69. It may here be remarked in passing that the principle of the wheel and axle is also employed in some forms of the cycle. The *power is applied to the crank* on which the foot rests, and the '*weight*' overcome is the resistance which the road offers to the free motion of the machine. Applying the principle of the 'wheel and axle,'

$\frac{W}{P} = \frac{\text{length of the crank}}{\text{radius of the front wheel}}$, and as the ratio



$\frac{\text{lenght of crank}}{\text{radius of front wheel}}$ is always less than unity, the machine works at a 'disadvantage,' *i.e.*, it is easier to push the machine on the road by the hand than by applying force at the crank, but *the object is to gain speed*, and this evidently is secured, as at every complete turn of the crank the front wheel moves over a length equal to its circumference. For instance, if the length of the crank be four inches and the radius of the wheel twenty-eight inches,

then the circumference of the wheel will be seven times the circumference of the circle described by the crank, and consequently at every turn of the crank, the bicycle will be carried over a distance equal to seven times the circumference described by the crank.

EXAMPLES.

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1. In a Chinese windlass the two axles are 1 foot and 2 feet respectively in thickness and the handle is 1 ft. 6 in. long. Find what power will support a weight of 1 cwt.?

The condition of equilibrium is

$$\frac{W}{P} = 2 \times \frac{\text{length of the handle}}{\text{difference between the radii of the two axles.}}$$

Now, the radii of the two axles are 6 inches and 12 inches respectively, and the difference between them is 6 inches; therefore, by substitution :—

$$\frac{112}{P} = 2 \times \frac{18}{6} = 6,$$

$$\text{and} \quad 6\,P = 112\,;$$

$$\text{hence} \quad P = 18\tfrac{2}{3} \text{ lbs.}$$

2. The two radii of a compound axle are 10 and 6 centimetres and the radius of the wheel is 6 decimetres: what is P when W is 300 kilogrammes? *Ans.* 10 kilos.

3. In the differential wheel and axle the diameter of the wheel is 5 ft. and the two portions of the axle have diameters 6 in. and 8 in. Find the power necessary to support a weight of 3 tons. *Ans.* 112 lbs.

4. By using a Chinese windlass a man is able to support a weight of 720 lbs. by a force of 12 lbs. The radii of the axles are 4 and 5 in. Find the length of the handle. *Ans.* 30 in.

5. By using the differential axle a man is able to support a weight of 112 lbs. by a force of 14 lbs. The handle is $\frac{2}{3}$ ft. long. Find the difference between the radii of the two axles. *Ans.* 2 inches.

6. In a bicycle, the circumference of the circle described by the crank is just 1 ft.; how far would the bicycle travel on level ground in one revolution, when the diameter of the wheel is 10 times the diameter of the crank? *Ans.* 10 ft.

QUESTIONS.

1. Describe the *wheel and axle.* How are the 'power' and the 'weight' applied?

2. In which way is it a modification of the lever?

3. Determine the condition of equilibrium of the *wheel and axle,* and state how this machine, unlike the lever, *always* acts at a mechanical advantage.

4. Describe some modifications of the *wheel and axle* and investigate the condition of equilibrium separately in each case.

5. Describe the Chinese windlass; why is it so called? Investigate its mechanical advantage.

CHAPTER IV.

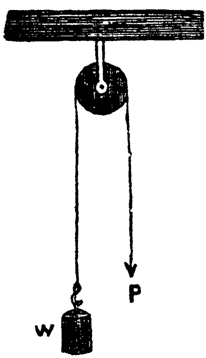
The Pulley.

70. **The pulley.**—The pulley is a small circular disc or wheel, the circumference of which is **grooved** to receive a **cord** which **passes over it.** This wheel, which is called the **sheaf**, is made to **revolve** freely **about an axis fixed in** a frame work called **the block.** The **weight hangs** at **one end** of the cord and the **power** is applied at the **other end.**

The tension in a string or the force with which a string is stretched is the same in all its parts and the tension is not altered by merely passing the string over a smooth wheel. (*Vide* art. 15.) *The mechanical principle involved in the pulley is this constancy of the force of tension in all parts of the same string.*

The *advantage* of having the *sheaf rotating* is that the cord slides easily on it.

71. **The single fixed pulley.**—As the tension in all parts of a cord is always the same, therefore, in a fixed pulley the **cord** is **stretched** equally on the two sides **by two equal forces,** the 'power' and the 'weight,' **in opposite directions,** and hence, *for equilibrium*, P=W.



Or, on the 'principle of energy,' if P is lowered through a distance d, W is raised through the same distance and the *equation of work* is $P\times d = \mathbf{W}\times d$,

$$\therefore\ \mathbf{P=W.}$$

No mechanical advantage is, therefore, gained by using a fixed pulley; however by its agency we can *raise a weight by means of a force not acting vertically upwards and thus apply the 'power' with greater convenience to ourselves*, as shown in the figure.

In a single fixed pulley in equilibrium,

(i) the **tension** in the cord=P=W, and

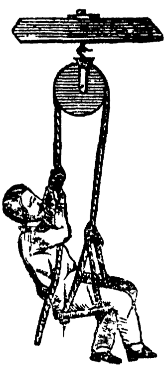
(ii) the **pressure on the beam** = P + W.

EXAMPLES.

1.—Indicate how by using a fixed pulley a man can haul himself up with a force equal to only a fraction of his own weight.

If a man just lifts himself up from the floor by holding fast to a cord suspended from the roof, his whole weight is supported by the hook in the ceiling and the *tension on the cord is equal to his weight* (*W*).

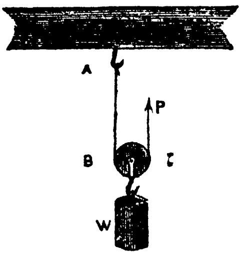
If on the other hand he holds fast to one end of a cord passing over a fixed pulley, he must attach a counter-balancing load (W) as great as his own weight to remain hanging in air. This may be done by attaching a small platform at the other end of the rope and then by putting weights on it. Obtaining equilibrium, if by a dexterous movement he next transfers himself also to the platform, all the time maintaining his hold and exerting the same downward pressure as before on the rope, the previous condition of equilibrium will not be changed and *a total weight of 2 W will be supported by a downward pressure of W* (*i. e., by a muscular exertion equal to the weight of the body*).



Now, if the man alone is on the platform, the weight to be supported is W and not 2W, and the downward pressure he has to exert to maintain equilibrium is only ½W. Therefore, a man can support his own weight by means of a fixed pulley and cord by sitting on a platform attached at one end of the cord and by exerting a force equal to half his own weight at the other end of the cord, and by a slightly greater exertion than half his weight he can haul himself up.

2. A man weighing 140 lbs. forces up a weight of 80 lbs. by means of a fixed pulley under which he stands; find his pressure on the floor. *Ans.* 60 lbs.

72. The single moveable pulley.—In a single moveable pulley we have a **weight W supported equally by the two** parallel **parts of the cord,** hence the tension of each of the parts is $\frac{1}{2}$ **W** and **half the weight is supported** by the hook **A** and the other half **by the power** applied at the other end ;



$$\therefore \quad P=\tfrac{1}{2}W, \text{ or } \frac{W}{P}=2.$$

Or, on the 'principle of energy,' when W rises through a distance *d*, to keep the string stretched P has to be moved through the distance $2d$ and the '*equation of work,*' therefore, is $W \times d = P \times 2d$,

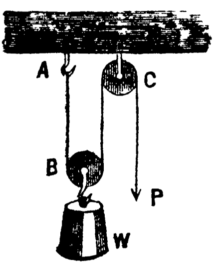
$$\text{and } \frac{W}{P}=2.$$

In a single moveable pulley

(i) the **tension** $=P=\frac{1}{2}W$;

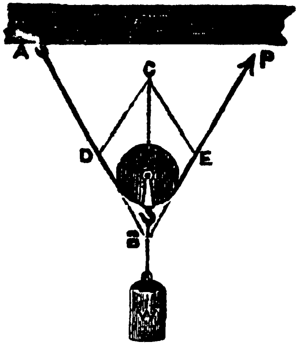
(ii) the **pressure on the beam** $=\frac{1}{2}$ W, because the other half of the weight is supported by P ;

(iii) if the weight of the pulley (w) be not neglected, then the tension in each part of the string is equal to $\frac{1}{2}(W+w)$ and $P=\frac{W+w}{2}$.



An extra *fixed* pulley may also be employed in conjunction with the moveable pulley ; this does not alter the condition of equilibrium, but only helps to change the direction in which the force or 'power' is applied.

If the cords be not parallel but inclined, as shown in the figure, then the condition of equilibrium will be different. For, produce the directions of the two parts of the cords to meet in B; in the vertical line through B take any length BC having as many units of length as there are units of weight in W, and from C draw CE parallel to AB, and CD to



BE. Then, as BC is in the line of action of W, BE in that of P and BD in that of the tension (T) in the left hand cord, and further, as BC also represents W in magnitude, therefore, by the 'parallelogram of forces,' the lines BC, BE and BD represent the forces W, P and T also in magnitude and P : W : : BE : EC;

hence, the **mechanical advantage**, $\frac{W}{P} = \frac{BC}{BE}$.

Now as the tensions in the two parts of a cord are equal, therefore, P=T and BE=BD=CE;

but in the triangle BCE,
BC is less than BE+EC, (Eu., I., 20.)
∴ BC is less than 2 BE,
and the value of $\frac{BC}{BE}$ is less than 2;

therefore, also that of $\frac{W}{P}$ **is less than 2**;

while when the cords are parallel $\frac{W}{P} = 2$.

This is true in every case, hence *when the cords are not parallel, the mechanical advantage is always less than when the cords are parallel.*

A reference to the figure will make it evident that we have here really to deal with three forces in equilibrium meeting at different angles at a point (B), and that they are properly represented in magnitude by the sides and diagonal of a parallelogram. Hence, in finding the numerical relation between these forces in special cases, we can apply the formulæ of article 23; but we must remember that W is the anti-resultant of P and T, and, therefore, *equal in magnitude* to R of the formulæ and further that on the 'principle of equality of tensions,' P=T.

(*a*) *For an inclination of* 30°,

$$\therefore \quad W^2 = P^2 + P^2 + P^2\sqrt{3},$$

$$\text{and} \quad W = P\sqrt{\left\{2 + \sqrt{3}\right\}};$$

(*b*) *for an inclination of* 45°,

$$W^2 = P^2 + P^2 + P^2\sqrt{2},$$

$$\text{and} \quad W = P\sqrt{\left\{2 + \sqrt{2}\right\}};$$

(*c*) *for an inclination of* 60°,

$$W^2 = P^2 + P^2 + P^2,$$

$$\text{and} \quad W = P\sqrt{3};$$

(*d*) *for an inclination of* 90°,

$$W^2 = P^2 + P^2,$$

$$\text{and} \quad W = P\sqrt{2};$$

(*e*) *for an inclination of* 120°,

$$W^2 = P^2 + P^2 - P^2,$$

$$\text{and} \quad W = P;$$

(*f*) *for an inclination of* 135°,

$$W^2 = P^2 + P^2 + P^2\sqrt{2},$$

$$\text{and} \quad W = P\sqrt{\left\{2 - \sqrt{2}\right\}};$$

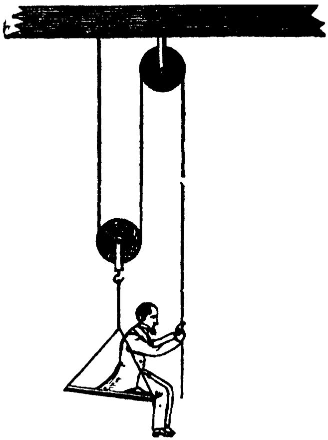
and (*g*) *for an inclination of* 150°,

$$W^2 = P^2 + P^2 - P^2\sqrt{3},$$

$$\text{and} \quad W = P\sqrt{\left\{2 - \sqrt{3}\right\}}.$$

EXAMPLES.

1. A man sitting upon a board suspended from a single moveable pulley pulls downwards at one end of a rope which passes under the moveable pulley and over a pulley fixed to a beam overhead, the other end of the rope being fixed to the same beam. What is the smallest proportion of his whole weight with which the man must pull in order to keep himself lifted up ?



The downward pressure on the platform is diminished by the muscular pull or power (P) exerted by the man. Again, as the tension in all parts of a cord is the same the upward tension in each of the 'strings' is equal to P. Now, as the *actual weight supported* on the platform is equal to W—P and opposed to it are the two equal forces P and P acting upwards through the medium of the two 'strings' on the two sides of the moveable pulleys therefore, for equilibrium

$$W - P = 2P$$

$$\text{or } P = \frac{W}{3}.$$

2. What force can sustain a weight of 59 lbs. by means of one movable pulley, the strings being parallel. *Ans.* 28 lbs.

3. A force of 20 lbs. acting upwards supports a weight by means of a single moveable pulley ; find the weight and the pressure on the fixed point of support. *Ans.* 40 lbs ; 20 lbs. respectively.

4. A weight of 30 lbs. is supported on a single moveable pulley, and the cord to which the power is applied passes over a fixed pulley, so that the three parts of the cord are parallel. Find the power and the strain on the beam. *Ans.* 15 lbs ; 45 lbs.

5. In a single moveable pulley, if the weight of the pulley be 8 lbs., find the force required to raise a weight of 42 lbs. *Ans.* 25 lbs.

6. In a single moveable pulley, if the weight of the pulley be 1 lb., find the force required to support a weight of 7 lbs. *Ans.* 4 lbs.

7. In a single moveable pulley, if the weight of the pulley be 4 lbs., find the weight that can be lifted by a power of 20 lbs. *Ans.* 36 lbs.

8. A force of 6 lbs. is exerted by each of the vertical parts of the string which passes round a single moveable pulley of weight 2 lbs.; what weight does the pulley carry? *Ans.* 10 lbs.

9. By using a single moveable pulley with cords parallel to a man, can lift a weight of 1 cwt. by exerting a force of 60 lbs. Find the weight of the pulley. *Ans.* 8 lbs.

10. If a man has to raise a weight, and has only one pulley at his disposal, show how he must apply it in order to obtain the utmost advantage.

11. If there be two strings at right angles to each other in a single moveable pulley, find the force which will support a weight of $\sqrt{2}$ lbs.

$$W^2 = P^2 + P^2$$

$$\therefore (\sqrt{2})^2 = 2P^2$$

$$\therefore 2P^2 = 2$$

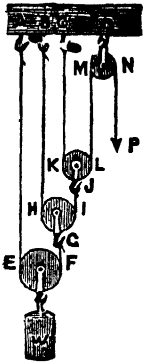
$$\therefore P = 1 \qquad \textit{Ans.}\ 1\text{ lb.}$$

12. In order that the power may be equal to the weight in the single moveable pulley, at what angle must the parts of the string be inclined to each other. *Ans.* 120°.

13. A weight of 30 lbs. is to be raised by means of a single moveable pulley. What is the power required when each string is inclined at an angle of 30° to the vertical. *Ans.* $10\sqrt{3}$.

73. Systems of pulleys.—Various combinations of pulleys are in use, which may all be classified under one of the **three** principal heads called **systems.**

THE FIRST SYSTEM.—In this system the **pulleys are all moveable.. Each** pulley has **a separate string, one end** of which is attached to a fixed point **in the beam, the other end** being **attached to** the hook of **the moveable pulley next above it.** The **power** is **applied at the free end of the last string** and the **weight is attached to** the hook of **the lowermost** moveable **pulley.**



Sometimes a fixed pulley is also attached to the beam (as in the figure), near the highest moveable pulley, round which the string of the last moveable pulley is passed. This is only done in order to change the direction of application of the power for convenience; its presence, therefore, does not alter the condition of equilibrium of the system.

Condition of equilibrium—(*If we discard the weights of the moveable pulleys,*) as W is equally supported by the tension of the strings AE and FG,

$$\text{tension of FG} = \frac{W}{2},$$

$$\text{similarly, tension on IJ} = \tfrac{1}{2} \text{ tension on FG} = \tfrac{1}{2}.\frac{W}{2} = \frac{W}{2^2},$$

$$\text{and tension on LM} = \tfrac{1}{2} \text{ tension on IJ} = \tfrac{1}{2}.\frac{W}{2^2} = \frac{W}{2^3},$$

but tension on LM $=$ P,

$\therefore P = \frac{W}{2^3}$ when there are *three* moveable pulleys, and similarly with n moveable pulleys

$$P = \frac{W}{2^n} \text{ or } 2^n P = W.$$

Or, on the 'principle of energy,' as it evident that to raise W through x inches, the cord of the lowest pulley must be shortened by $2x$ inches, that of the one above by $(2\times 2x)$ or 2^2x inches, that of the next $2(2^2x)$ or 2^3x inches, and of the n^{th} or uppermost moveable pulley by 2^nx inches, P has also to be moved through 2^nx inches;

$\therefore$ *the equation of work*, P. d = W. d', becomes

$$P.\ 2^n x = W.\ x$$

$$\text{and } W = 2^n P.*$$

The **Pressure on the beam**, when there is a fixed pulley attached (as in the figure), is equal to W + P, because both W and P pull against the beam. This, by substitution, is equal to $2^nP + P = (2^n + 1)P$; but without it, the weight will be evidently supported by the beam *and* the power, and the pull on the beam will be

$$W - P = 2^n P - P = (2^n - 1) P.$$

* **If the weights of the moveable pulleys** be not disregarded, then beginning with the lowermost pulley, if their weights be x, y, z, respectively,

$$\text{tension on FG} = \tfrac{1}{2}(W + x),$$

$$\text{tension on IJ} = \tfrac{1}{2}\text{ tension on FG} + \tfrac{1}{2}y,$$

$$= \frac{1}{2^2}(W + x) + \tfrac{1}{2}y,$$

$$\text{tension on LM} = \tfrac{1}{2}\text{ tension on IJ} + \tfrac{1}{2}z,$$

$$= \frac{1}{2^3}(W + x) + \frac{1}{2^2}y + \tfrac{1}{2}z;$$

$$\text{but tension on LM} = P,$$

$$\therefore P = \frac{W}{2^3} + \frac{x}{2^3} + \frac{y}{2^2} + \frac{z}{2}.$$

If the weight of each pulley be the same, say w,

$$\text{then } P = \frac{W}{2^3} + \frac{w}{2^3} + \frac{w}{2^2} + \frac{w}{2} = \frac{W}{2^3} + w\left(1 - \frac{1}{2^3}\right),$$

and similarly if n be the number of moveable pulleys

$$\text{then } P = \frac{W}{2^n} + w\left(1 - \frac{1}{2^n}\right)$$

$$\text{or } P - w = \frac{W - w}{2^n}.$$

EXAMPLES.

1. In a 'system' where there are 7 moveable pulleys hung by separate strings which are all parallel, what power must be applied to support a weight of 384 lbs.?

The *first system* is usually described as the system in which the pulleys are hung by separate strings."

By first principles :—As $W = 384$ lbs.,
the pull on the 2nd pulley (from the

$$\text{bottom}) = \frac{384}{2} \text{ lbs.}$$

$$\text{,, ,, on the 3rd ,,} \quad = \tfrac{1}{2}.\frac{384}{2} = \frac{384}{2^2} \text{ lbs.,}$$

$$\text{,, ,, on the 4th ,,} \quad = \tfrac{1}{2}.\frac{384}{2^2} = \frac{384}{2^3} \text{ lbs.,}$$

$$\text{and on the 7th ,,} \quad = \tfrac{1}{2}.\frac{384}{2^5} = \frac{384}{2^6} \text{ lbs.,}$$

$$= \frac{384}{64} = 6 \text{ lbs.,}$$

The tension, therefore, on the hook of the uppermost (7th) moveable pulley is equal to that of 6 lbs. This is equally divided between the beam-hook and the power,

$$\therefore P = 3 \text{ lbs.}$$

2. In the first system of pulleys, what will be the power if the weight be 16 lbs., the number of moveable pulleys being 3? *Ans.* 2 lbs.

3. If three moveable pulleys are arranged according to the first system, find what weight will be supported by a power of 20 lbs.

(*b*) If the same number of pulleys is arranged according to the third system, calculate what weight will be sustained by the same power.

(*a*) *By first principles* :—

The weight supported by the cord of the lowermost pulley

$$= W,$$

$$\therefore \text{ the pull on the second pulley } = \frac{1}{2} W,$$

$$\text{the pull on the third pulley } = \frac{1}{2}.\frac{W}{2} = \frac{W}{4},$$

and this is equally divided between the beam-hook of the last moveable pulley and the power.

$$\text{Therefore } P = \frac{1}{2} \cdot \frac{W}{4} \text{ or } \frac{W}{8},$$

$$\text{but } P = 20 \text{ lbs.},$$

$$\therefore \frac{W}{8} = 20,$$

$$\text{and } W = 160 \text{ lbs.}$$

(*b*) *By first principles* :—

Pull on the weight-bar due to P and transmitted through the first string	= 20 lbs.
Pull on the bar due to the second string	= 40 lbs.
Pull on the bar due to the third string	= 80 lbs.
Total pull = 20 + 40 + 80 lbs.	= 140 lbs. = W.

4. How many pulleys (weightless) must there be in the system, where each pulley hangs by a separate cord, in order that 1 lb. may support 128 lbs.? *Ans.* 7 moveable.

5. In the first system, a weight of 640 lbs. is supported by a power of 5 lbs. what is the number of moveable pulleys? *Ans.* 7.

6. In a system of three weightless moveable pulleys, a man sits in a chair which hangs from the lowest pulley, and supports himself by holding the uppermost cord of the system, which passes over a fixed pulley. If the man weighs 160 lbs., find the tension of the cord which he holds, and the pull exerted on the lowest pulley.

Let x denote the tension of the cord *i. e.* the power.

$\therefore$ The pull on the lowest pulley or weight $= 160 - x$.

$$W = 2^n P.$$

$$\therefore 160 - x = 2^3 x$$

$$\therefore 9x = 160.$$

$$x = 17\tfrac{7}{9}$$

$\therefore \quad P = 17\frac{7}{9}$; $W = 160 - 17\frac{7}{9} = 142\frac{2}{9}$. *Ans.* $17\frac{7}{9}$; $142\frac{2}{9}$ lbs.

7. A man weighing 160 lbs. raises 4 cwt. by a system of four moveable pulleys of negligible weight arranged according to the first system; and the cord which he holds passes over a fixed pulley above his head. Find the tension on the power-end of the cord and the pressure on the ground. *Ans.* 28 lbs.; 132 lbs. respectively.

8. In the preceding example, show that the greatest weight which the man can support without being lifted off his feet by the cord on which he pulls is $1\frac{1}{7}$ tons.

9. A man weighing 70 kilogrammes raises a weight of 800 kilogrammes by a system of four moveable pulleys, arranged according to the first system, what is his pressure on the floor on which he stands?

Ans. 20 kilos.

10. If a man supports a weight equal to his own, and there are three moveable pulleys arranged according to the first system, find the pressure on the floor on which he stands. *Ans.* $\frac{9}{8}$ of his weight, if he pulls upwards; $\frac{7}{8}$ if he pulls downwards by means of a fixed pulley.

11. Which is more effective machine, a wheel and axle whose radii are in the proportion of 20 : 3, or an arrangement of pulleys of the first system in which there are three moveable pulleys? *Ans.* The pulleys.

12. Four pulleys are arranged according to the First System: the weight of each pulley is 1 lb.; what power will sustain a weight of 95 lbs.?

By *first principles* the weight supported by the cord of the lowermost pulley is $95 + 1 = 96$ lbs.

The pull on the 2nd pulley from the bottom $= \frac{96}{2} = 48$ lbs.,

the pull on the 3rd pulley $= \frac{48 + 1}{2} = \frac{49}{2} = 24\frac{1}{2}$ lbs.,

and the pull on the 4th or uppermost pulley $= \frac{24\frac{1}{2} + 1}{2} = 12\frac{3}{4}$ lbs.

This pull of $12\frac{3}{4}$ lbs. is equally divided between the beam-hook of this last moveable pulley and the Power,

$$\therefore P = 6\tfrac{3}{8} \text{ lbs.}$$

13. If the weights of the pulleys in the first system, commencing with the highest, be 1, 2, 3, 5 lbs. respectively, find what power will sustain a weight of 25 lbs. *Ans.* $3\frac{1}{4}$ lbs.

14. Four pulleys, whose weights, beginning with the highest, are 3, 4, 2 and 6 lbs., respectively, are arranged according to the first system, what power will sustain a weight of 462 lbs.? *Ans.* 32 lbs.

15. If three moveable pulleys, the weights of which are 2 ozs., 4 ozs., and 8 ozs., be arranged as in the first system, what is the least force that will raise a weight of 104 ozs.? *Ans.* 16 ozs.

16 If four pulleys, weighing, respectively, 3, 7, 4, and 9 lbs., are arranged according to the first system, which is the most and which the least advantageous arrangement ?

Ans. { (1) The lowest, 9, the next 7, the third 4, and the highest 3.
(2) „ „ 3 „ „ 4 „ „ 7 „ „ „ 9.

17 What weight will be sustained by a power of 100 lbs. in the most advantageous arrangement of 4 pulleys, according to the first system, the pulleys weighing respectively 4, 8, 4, and 10 lbs. ? *Ans.* 1526 lbs.

18. If in the first system there be four moveable pulleys, the weight of each of which is 1 lb., find the power, when the weight supported is 33 lbs. Also find the whole strain on the fixed beam. *Ans.* P = 3 lbs. and pressure = 34 lbs. when P acts upward, and 40 lbs. when P acts downwards.

19. Find the force exerted on the fixed beam from which the pulleys are hung, when a weight of 112 lbs. is supported by means of three moveable pulleys, whose weights beginning from the lowest are 2 lbs, 1 lb. and 1 lb. *Ans.* 101 lbs.

20. A man weighing 10 stones supports a weight of 91 lbs. by means of 3 moveable pulleys arranged in the first system and weighing, respectively, 2 lbs., 4 lbs., 5 lbs. beginning with the highest. What is the thrust of the man on the ground ? *Ans.* 9 stones.

21. If a weight of 80 lbs. be supported by a force of $5\frac{15}{16}$ lbs. in a system of 4 equal moveable pulleys arranged according to the first system, find the weight of each pulley. *Ans.* 1 lb.

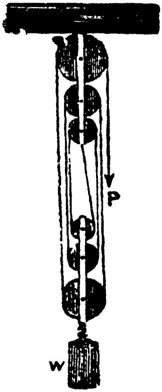
22. If the force on the fixed beam be 117 lbs., the weight to be supported 115 lbs. and there be four equal moveable pulleys, find the power and the weight of each pulley. *Ans.* P = 10 lbs ; $w = 3$ lbs.

23. How many pulleys at least must be employed in the first system in order that 1,500 lbs. may be raised by 30 lbs., if each pulley weighs 4 lbs.? *Ans.* 6.

24. Find the relation between the power (P) and the weight (W) in a system of 5 moveable pulleys, in which each pulley hangs by a separate string and the weight of each pulley is equal to P. *Ans.* $P = W$.

25. In the first system, prove that there will be equilibrium, if the power, the weight, and the weight of each moveable pulley is the same.

THE SECOND SYSTEM.—In this system of pulleys there are **two 'blocks'** or sets of pulleys; on the **lower moveable** block the **weight hangs** and the **same string passes round all** the pullys successively. The string is nsually first attached to the hook of the upper of fixed block.



Another arrangement for this system is given in the second figure; the pulleys in each block are *placed side by side* and rotate about a common axis, the advantage being that it *allows the weight to be raised to a greater height* before the blocks meet.

Condition of equilibrium.—(Discarding the weight of the blocks) it is clear that on the principle of *equality of tension in the same cord*, there are three pulleys in each block or six pulleys in all (as in the figures), the weight will be equally supported by the six 'strings,' and the power applied at the end of the cord must, therefore, be only half of the weight; and similarly with a total number of n pullyes in tha two blocks together, there will be n 'strings' and

$$\mathbf{P} = \frac{\mathbf{W}}{\mathbf{n}} \text{ or } \mathbf{W} = \mathbf{n}\,\mathbf{P}.$$

Or, on the 'principle of energy,' as it is evident that to raise W through x inches each 'string' must be shortened through that length, therefore, with a total number of n pulleys and n 'strings,' the end to which P is attacbed must be moved through nx inches.

Hence, the *equation of work* becomes

$$W.\,x = P.\,nx,$$

$$i.\,e.,\ \mathbf{P} = \frac{\mathbf{W}^*}{\mathbf{n}} \text{ or } \mathbf{W} = \mathbf{n\,P}.$$

If the cord is in the first instance attached to the top of the lower block, then with a total number of n pulleys there will be $n + 1$ strings and $P = \frac{W}{n+1}$.†

The **pressure on the beam** is equal to $\mathbf{W} \div \mathbf{P}$, because both W and P act in the same direction and pull against the beam.

EXAMPLES.

1. In the second system, the weights of the pulleys being neglected, if there be ten strings to the lower block, what power will support a weight of 1,000 lbs. ?

By first principles:—As there are ten 'strings,' the tension each string is one-tenth the weight,

∴ the power applied at the end of the tenth string must be equal to

$\frac{1}{10}$ of 1,0C0, or 100 lbs.

2. In the system of pulleys where the same string passes round all the pulleys, of which there are five at the lower block, find what power will support a weight of 1,000 lbs. *Ans.* 100 lbs.

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* **If the weight of the moveable block** be not disregarded, then as the weight of the whole block acts in the same direction as W, we must add the weight of the block w to W and $P = \frac{W+w}{n}$ or $\frac{W+w}{n+1}$, as the case may be.

† The student must draw the figure to satisfy himself on the point.

3. In the second system there are 4 pulleys in each block, and a weight 72 lbs. to be supported: required the power, (1st) when the end of the strings starts from the upper block, and (2nd) when it starts from the lower. *Ans.* (1) 9 lbs.; (2) 8 lbs.

4. What force is necessary to raise a weight of 120 kilogrammes by an arrangement of six pulleys in which the same string passes round each pulley. *Ans.* 10 kilos.

5. If in the preceding problem, the weight of the lower block be 6 kilogrammes, what additional force will be required? *Ans.* 0·5 kilos.

6. If there are 4 strings at the lower block in the second system, find the greatest weight which a man weighing 8 stones can possibly support. *Ans.* 4 cwts.

7. In the second system, what weight can be supported if there are 3 pulleys at the lower block, one end of the string being fastened to the upper block, and the weight of the lower block being three times the power exerted? *Ans.* $W = 3\ P$.

8. What power must a man weighing 150 lbs. exert to raise himself by a pair of pulley-blocks, each containing two wheels? *Ans.* $37\frac{1}{2}$ lbs.

9. Find the number of strings at the lower block in order that a power of 8 lbs. may support a weight of 1 cwt., the second system of pulleys being employed. *Ans.* 14.

10. Find the number of strings at the lower block (in the second system) in order that a power of 4 lbs. may support a weight of 40 lbs. *Ans.* 10.

11. Find the number of strings at the lower block in the second system in order that a power of 4 ounces may support a weight of 4 lbs. *Ans.* 16.

12. In the system in which the same string passes round all the pulleys a weight of 15 lbs. supports a weight of 80 by six pulleys, three being in each block; find the weight of the lower block and strain on the fixed point of support. *Ans.* 10 lbs.; 105 lbs.

13. A man supports a weight equal to half his own weight, by a system of pulleys, in which the same string passes round all the pulleys, the upper block being attached to the ceiling: if there be 7 strings at the lower block, find his pressure on the floor on which he stands. *Ans.* $\frac{13}{14}$th of his own weight.

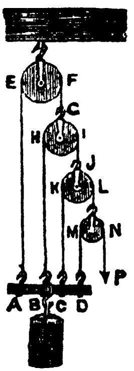
14. A man weighing 10 stones stands on the ground and pulls up 2 cwts, by an arrangement of pulleys as in the second system. There are 6 cords on the lower block. Find the pressure which he exerts on the ground and the pressure on the beam to which the upper pulley block is attached. *Ans.* $7\frac{1}{3}$ stones; $18\frac{2}{3}$ stones.

15. A man of weight 160 lbs. sitting in a loop of a rope which hangs from the lower block, raises himself by pulling down the loose end of the cord which passes round the pulleys. If the lower block weights 8 lbs. and there are 8 cords on this block, find the force with which he pulls *Ans.* Greater than $18\frac{2}{3}$ lbs.

16. In the second system if the weight of the lower block and the power be each p kilogrammes and the weight attached to the lower block be w kilogrammes, show that w is some odd or even multiple of p, according as the end of the string is fastened to the upper or lower block.

17. In the second system of pulleys show that if P descend through 1 ft., W will rise through $\frac{6}{n}$ inches, where n is the number of pulleys in the lower block.

THE THIRD SYSTEM.—In this system of pulleys, the **string** which passes round any pulley except the lowest has **one end** attached **to the weight** and **the other** end attached **to the hook of the next** lower pulley, while the string which passes round the lowest pulley has one end also attached to the weight but the other end supports the power. The **highest** pulley is **fixed,** while the **rest** are **moveable.**



Condition of equilibrium.—(Discarding the weights of the pulleys), it may be observed, that in a system of *four* pulleys, as shown in

the figure, the weight is supported by the tensions of the four strings DM, CK, BH, AE.

Now tension in DM $=$ P,

„ „ CK $=2$ P,

„ „ BH $=2$*ce*. 2 P $=4$ P,

and „ „ AE $=2$*ce*. 4 P $=8$ P ;

$\therefore$ W $=$ P $+2$ P $+4$ P $+8$ P,

$=15$ P $=(2^4-1)$ P.

Similarly, with n pulleys,

$$\mathbf{W}=(2^n-1)\,\mathbf{P},$$

$$\text{and } \mathbf{P}=\frac{\mathbf{W}}{2^n-1}.$$

Or, *on the 'principle of energy,'* it is evident that if W rises through x inches, the moveable pulley first from the top will descend through x inches, the second moveable pulley will descend through x inches owing to the rise of W and through another $2x$ inches owing to the motion of the pulley above, in all through $2x+x$ or $(2+1)\,x$ inches, similarly the third and last moveable pulley will descend through $x+2\,(2+1)\,x$ inches, *i.e.*, through $(2^2+2+1)\,x$ inches and the end at which P acts will similarly descend through $x+2\,(2^2+2+1)\,x$ inches, *i. e.*, through $(2^3+2^2+2+1)\,x$ inches ; and if there be n pulleys, the nth pulley will descend through $(2^{n-2}+\ldots+2^2+2+1)\,x$ inches, and the end at which P acts will descend through

$$x+2\,(2^{n-2}+\ldots+2^2+2+1)\,x \text{ inches,}$$

$$=(2^{n-1}+\ldots+2^3+2^2+1)\,x \text{ inches,}$$

$$=x\,(2^n-1) \text{ inches ;}$$

therefore, the *equation of work* becomes

$$x\,W = x\,(2^n-1)\,P,$$

$$i.\,e.,\ W = (2^n-1)\,P.*$$

The **pressure on the beam** $= W + P$, as both the power and the weight pull against the beam.

If in an arrangement on the first system, we discard the fixed pulley, which is not absolutely necessary, then it will be observed that *the third system is equivalent to the first turned upside down* and *vice versa*, the beam and W interchanging places.

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* If the weights of the moveable pulleys be not disregarded, then beginning with the lowermost pulley, if their weights be x, y, z, respectively

$$\text{tension in } DM = P$$

$$\text{,, \quad ,, } CK = 2\,P + x,$$

$$\text{,, \quad ,, } BH = 2\,(2P + x) + y,$$

$$= 2^2\,P + 2\,x + y,$$

$$\text{,, \quad ,, } AE = 2\,(2^2\,P + 2\,x + y) + z,$$

$$= 2^3\,P + 2^2\,x + 2\,y + z;$$

and $W = 2^3\,P + 2^2\,P + 2\,P + P + 2^2\,x + 2\,x + x + 2\,y + y + z,$

$$= (2^3 + 2^2 + 2 + 1)\,P + (2^2 + 2 + 1)\,x + (2 + 1)\,y + z;$$

If the weight of each pulley be the same, say w,

$$\text{then } W = (2^3 + 2^2 + 2 + 1)\,P + (2^3 + 2^2 + 2 + 2 - 3)\,w,$$

$$= (2^4 - 1)\,P + (2^4 - 5)\,w;$$

and if n be the number of pulleys

$$W = (2^n - 1)\,P + (2^n - n - 1)\,w.$$

EXAMPLES.

1. If in the third system of pulleys (the weight of each pulley being supposed zero) the number of pulleys be 6, what weight can be supported by a power of 12 lbs ?.

By first principles:—Commencing with the lowermost pulley,

Pull on the weight-bar due to the power and transmitted through the first string = P = 12 lbs.

Pull on the bar due to the second string	= 2. 12	„
„ „ „ „ third „	= 2. 24	„
„ „ „ „ fourth „	= 2. 48	„
„ „ „ „ fifth „	= 2. 96	„
„ „ „ „ six or last string	= 2. 192	„

and the total pull on the bar due to the weight.

$$= 384 + 192 + 96 + 48 + 24 + 12 \text{ lbs.}$$

$$\therefore W = 756 \text{ lbs.}$$

2. If three pulleys are arranged according to the third system, calculate what weight will be sustained by a power of 20 lbs.

Ans. 140 lbs.

3. If there be four moveable pulleys in the system in which all the strings are attached to the weight, find what power will support a weight of 155 lbs. *Ans.* 5 lbs.

4. In the third system with 2 moveable pulleys a certain power balances a weight of a lbs. If 5 lbs. be added to the power, what additional weight ean be raised. *Ans.* 35 lbs.

5. A weight of 210 lbs. can be supported in the third system by a power of 30 lbs.; find the number of pulleys. *Ans.* 3.

6. If the mechanical advantage in the third system be 63; how many moveable pulleys are there? *Ans.* 5.

7. A man weighing 3 cwts. holding the bar to which the strings are attached in the third system of pulleys, can just raise a child weighing $22\frac{2}{5}$ lbs. holding on to the last string. How many pulleys are there? *Ans.* 3 moveable pulleys.

8. A weight of 508 lbs. is hung on to a system of three pulleys of the third kind; the weights of the three pulleys are 8 lbs., 6 lbs., and 4 lbs., the last being the lowest. What power will be necessary to support the weight?

Commencing with the lowermost pulley,—

the pull on the weight-bar due to the power and transmitted through the first string $= P$ lbs.,

the pull on the bar through the second string $= 2P + 4$ lbs.,

the pull on the bar through the third string $=$

$$2(2P + 4) \div 6 = 4P + 14 \text{ lbs.},$$

and as the *total pull* on the weight-bar keeps the weight of 508 lbs. in equilibrium,

$$\therefore P + 2P + 4 + 4P + 14 = 508 \text{ lbs.}$$
$$7P = 490 \text{ lbs.}$$
$$\therefore P = 70 \text{ lbs.}$$

(*Note.*—The weight of the uppermost pulley is not taken into account, as that pulley is fixed).

9. Find the power necessary to support a weight of 50 lbs. with three moveable pulleys, arranged according to the third system, each pulley weighing 1 lb. *Ans.* $2\frac{3}{8}$ lbs.

10. If there are four moveable pulleys in the third system, and each weighs 2 lbs., what weight can be raised by a power equal to the weight of 20 lbs. *Ans.* 6 cwts.

11. In a third system of pulleys, if the weights of the pulleys are 1 lb., 2 lbs., 3 lbs. find the greatest weight and the least weight which can be kept in equilibrium by a power of 7 lbs., it being understood that the pulleys may be arranged in any order and that only two of them are moveable. *Ans.* 60, 54 lbs.

12. If there be three pulleys in the third system the weight of each of which is equal to the power show, that the power will support a weight 11 times as great as itself.

13. If there be three moveable pulleys, all of the same weight, find what this weight must be, in order that a power of 3 lbs. may support a weight of 50 lbs. *Ans* $\frac{5}{11}$ lbs.

14. If there are two moveable pulleys, of the same weight, find what this weight must be when the weight to be raised is 76 lbs. and the resultant force on the fixed pulley is 89 lbs. What is the power? *Ans.* $1\frac{1}{3}$ lbs.; 10 lbs.

15. In the third system of eight pulleys, find the ratio that the weight of each pulley must bear to the weight in order that the latter may just be supported by the total weight of the pulleys alone. *Ans.* 1 : 247.

74. Uses of the various systems.—Among the three systems mentioned above, the third and last system is of no practical use, as the strings always get entangled in actual practice. Of the remaining two systems, the 'first' gives a greater mechanical advantage than the 'second'; for on comparing formulæ, it will be found that in the first system each additional pulley doubles the advantage instead of merely increasing it by unity as in the second system. In the first system, however, the strings require to be attached much higher up than the point to which the weight has to be raised, because when the top pulley touches the beam, the weight is still lower down; this inconvenience is avoided in the second system and particularly in that form in which the pulleys are placed side by side in each block. The second system is, therefore, by far the most employed in practice.

MISCELLANEOUS EXAMPLES.

1. How would you arrange four pulleys for the support of a weight so as to obtain the greatest possible mechanical advantage from their use?

The *Mechanical advantage* of a machine is measured by the value of $\frac{W}{P}$, therefore, the advantage is the greatest possible when $\frac{W}{P}$ has the largest value.

Discarding the weights of the pulleys:—

(i) *in the first system* $W = 2^n P$,

$$\therefore \frac{W}{P} = 2^4 = 16;$$

(ii) *in the second system* $W = n P$,

$$\therefore \frac{W}{P} = 4;$$

(iii) and *in the third system* $W = (2^n - 1) P$,

$$\therefore \frac{W}{P} = 2^4 - 1 = 15.$$

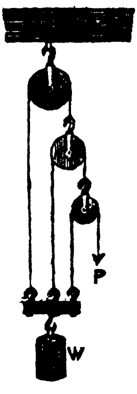
For a given value of n, therefore, the ratio $\frac{W}{P}$ or the mechanical advantage is greatest in the *first system.*

2. Make careful sketches of, (1) a system of weightless pulleys in which 1 lb. balances 32 lbs. and (2) a system of weightless pulleys in which 1 lb. balances 15 lbs., taking care in each case that the number of pulleys is the least possible.

3. Make a careful drawing of a system of weightless pulleys in which a force equal to the weight of 1 lb. can support a weight of 31 lbs.

4. Sketch a system of pulleys, one fixed and two moveable, in which P is applied to one end of the string passing round each pulley, the other end being attached to one moveable block and W hanging from the other block.

5. Arrange combination of pulleys by which a man weighing 150 lbs. may be able, by pulling downwards, to lift a weight of 10 cwts.



6. Find the correct position of the weight W in the figure, so that the rod on which it hangs may be horizontal; the three hooks on the rod being equidistant.

(The position of the weight in the figure is not quite correct.)

The upward pull exerted by the first string from the right hand = P, the upward pull exerted by the second or middle string = 2 P, and the upward pull exerted by the third string = 4 P.

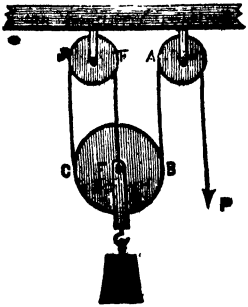
The resultant, therefore, of these three parallel forces is equal to 7 P (art. 31), and the anti-resultant W must act vertically downwards at the same point as the resultant 7 P to keep the rod horizontal.

Let l be the length of the rod, then taking moments round the right-hand end where the first string is attached,

$$(P + 0) + (2P \times \tfrac{1}{2}l) + (4P \times l) = 7P \times x \text{ and } x = \tfrac{5}{7}l;$$

i.e., the resultant acts at a distance from the right-hand end equal to $\frac{5}{7}$ of the length of the rod, and therefore, to keep the rod horizontal, the anti-resultant or weight W must also be hung vertically down from the same point.

7. Find the relation between P and W in a system of one moveable and two fixed pulleys arranged as shown in the annexed diagram.



There is a single cord passing round all the three pulleys and the moveable pulley with the weight hanging from it is supported by three parallel strings therefore, on the principle of 'equality of tensions in a string,'

tension in BA = tension in CD = tension in EF,

and the pulley (w) and the weight (W) are supported by these three equal tensions,

$$\therefore W + w = 3ce \text{ tension in BA},$$

$$\text{but tension in BA} = P,$$

$$\therefore W + w = 3P.$$

If the weight of the pulley be neglected,

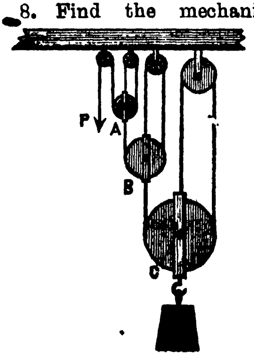
$$\text{then } W = 3P.$$

Or, on the 'principle of energy,' to raise W through x inches, each of the three cords supporting it must be shortened through x inches and P must descend through a distance $3x$ inches and the *equation of work* becomes

$$Wx = 3Px$$

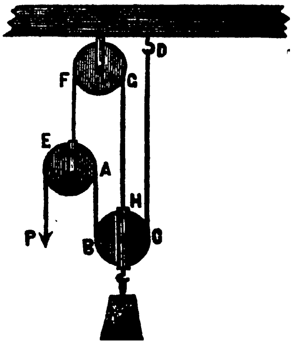
$$\text{and } W = 3P.$$

8. Find the mechanical advantage of a system of pulleys arranged as in the annexed figure, the weight of the moveable pulleys being neglected.



From the preceding example, it is evident that P will support a weight equal to 3 P at A, or that there will be an upward pull of intensity 3 P at A; this in its turn will cause an upward pull of *thrice* 3P or 9P at B and this last in its turn again will cause an upward pull of *thrice* 9P or 27P at C; thus the weight supported at C by the power P will be equal to 27 P,

$$\text{or } W = 27P \text{ and } \frac{W}{P} = 27.$$



9. Find the relation between W and P in a system of pulleys arranged as in the annexed diagram, supposing the weight of each moveable pulley is w.

(This arrangement is often called the *Spanish Burton.*)

$W + w$ is supported by the tension in C D + tension in H G + tension in B A, ... (i),

and on the principle of 'equality of tension,' tension in C D = tension in B A = P; and tension in H G = tension in E F

$= P +$ tension in B A $+ w = 2P + w$;

hence by substitution in (i)

$$W + w = P + (2P + w) + P;$$

$\therefore W = 4P$ and the weights of the two moveable pulleys have no effect as they are equal.

QUESTIONS.

1. Describe a single *pulley* and name its several parts.

2. What is the main principle underlying the construction of a pulley? What is the advantage of the sheaf rotating?

3. When is a pulley called *fixed* and when *moveable*?

4. What is the advantage of using a *fixed* pulley?

5. What mechanical advantage can you get from a *single moveable* pulley? Why is it not by itself much employed in practice?

6. Sketch an arrangement of a fixed and a moveable pulley for lifting up weights. Discuss its 'mechanical advantage,' and state why it is to be preferred to a single moveable pulley.

7. Compare the loads that could be kept in equilibrium on two moveable pulleys; in one the cords being parallel as usual and in the other inclined at a definite angle.

8. Write down the values of $\frac{W}{P}$ for a single moveable pulley with cords inclined at angles of 30,° 45,° 60,° 90,° 120,° 135° and 150,° and deduce them.

9. What are the *systems* of pulleys? How many are they? Describe them and note the distinguishing features of each.

10. Deduce the value of $\frac{W}{P}$ for each of these systems. Also determine the pressure on the beam-hook in each case.

11. In which system would a given number of pulleys be utilised with maximum advantage and in which with minimum of advantage?

12. If the weights of the pulleys are also taken into consideration, show how the condition of equilibrium is changed (*a*) in a single moveable pulley and (*b*) in each of the three systems.

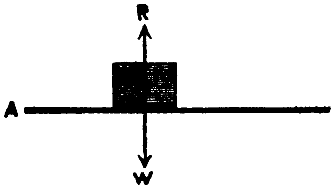
13. Which 'system' of pulleys is most employed in practice; what is the reason? What particular form of that system is generally preferred, and why?

14. State the various uses for which a set of pulleys may be employed.

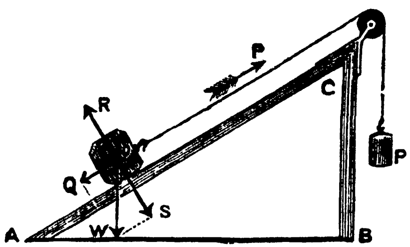
CHAPTER V.

The Inclined Plane.

75. The Inclined Plane.—If a body rests on a horizontal surface, then the weight of the body (W) causes a vertical downward pressure on the surface and the plane on which it rests exerts a re-action (R) in the line of action of the pressure, *i.e.*, at right angles to the surface. These forces (W and R) being equal and opposite, the body remains at rest.



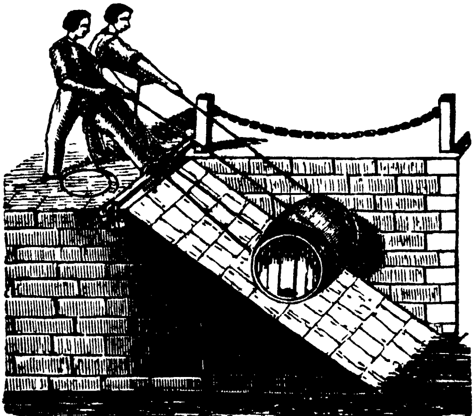
But when a body is placed on a smooth inclined surface, the whole force of gravity does not produce pressure on surface ; one component only exerts pressure, while the other component produces motion down the plane. In the figure, if we resolve the force of gravity acting on the body, *i.e.*, W, into two components, *viz.*, S at right angles to the surface and Q parallel to the surface, then the component S produces pressure on the plane at right angles to its surface, which is counteracted by the (equal and opposite) re-action of the plane (R), while the component Q *remains uncompensated.* Q will, therefore, drag the body down the plane, and to prevent this, it is necessary to apply an equal and opposite force P to the body.



Hence, on an inclined plane when a body is in equilibrium there are **three forces acting** on the body, *viz.*, (1) the **weight** (W) of the body, (2) the **power** (P), *i.e.*, the force required to prevent the body from sliding down the plane, and (3) the **re-action** (R) of the plane ; a mass may thus

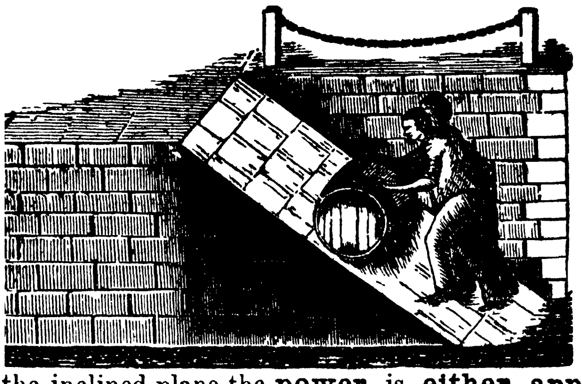
be apparently supported by a **force less than** its own **weight**, the reason being that the rest of the necessary force is supplied by the re-action of the plane.

Def.—A rigid plane inclined at an angle to the horizon is called an **inclined plane.**



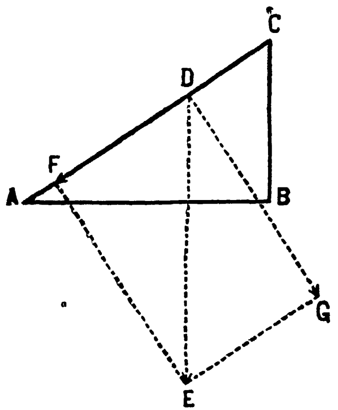
The surface **A** C is the **length** of the plane, AB, is the **base**, and BC at right angles to AB is the **height** of the plane.

The angle BAC is called the **inclination** of the plane; the slope is also frequently measured by the ratio of the height to the length; for example, if BC be the tenth part of AC, the plane is said to have an inclination or **gradient** of one in ten or to **rise** one in ten.



On the inclined plane the **power** is **either applied parallel to the plane or** parallel to the **base**, as shown in the two figures.

76. Condition of equilibrium of a body resting on smooth inclined plane and acted on by a **force parallel to the plane.**



Let ABC be an inclined plane and let the magnitude and direction of a **weight** placed on it, say at D, be **represented** by the **vertical** line **DE** in length equal to **AC**, the 'length' of the plane.

Then, if by the 'parallelogram of forces' we resolve DE into two components at right angles to one another (art. 19) the **power** will be **represented by DF** parallel to the plane, and the **pressure** on the plane **by** the line **DG**.

Now, as DE is taken equal to AC, and as angle DGE = angle ABC (both being right angles),

∴ angle DEG is equal to angle ACB, and angle EDG to angle BAC,

and triangle CDF = triangle ABC in every respect,

and DF = BC = height of the plane;

also DG = AB = base of the plane.

Therefore, W : P : Pr :: $l : h : b$,

$$\text{and } \frac{\text{W}}{\text{P}} = \frac{\text{length of the plane}}{\text{height of the plane}}.$$

Or on the 'principle of energy,'

$$\text{W} \times d = \text{P} \times d',$$

and if the weight be raised all the way from A to C up the inclined plane, it is practically *raised against the force of gravity through the height BC only*; to accomplish this the power must move through a distance equal to AC, which is the length of the plane. The *equation of work*, therefore, becomes

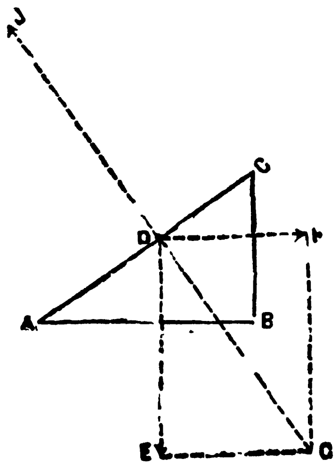
$$W \times \text{height of the plane} = P \times \text{length of the plane},$$

$$\textit{i.e.,}\ \frac{W}{P} = \frac{\text{length of the plane}}{\text{height of the plane}}.$$

Note.—Again, as W : Pr :: *l* : *b*,

$$\therefore \left\{\begin{matrix}\text{Pressure} \\ \text{on the plane}\end{matrix}\right\} : W :: \left\{\begin{matrix}\text{base of} \\ \text{the plane}\end{matrix}\right\} : \left\{\begin{matrix}\text{length of} \\ \text{the plane}\end{matrix}\right\}$$

77. Condition of equilibrium on an inclined plane when **P** acts **parallel to the base.**



Let ABC be an inclined plane and let the magnitude and direction of a **weight** placed on it, say at D, be **represented** by the vertical line **DE** in length equal to AB, and let **DF** be the **force** parallel to the base and **DJ** the **re-action** of the plane.

Now as W, P and R are in equilibrium and R is between W and P, therefore, by the 'parallelogram of force,' the resultant of W and P (DG in the figure) is equal in effect though opposite in direction to the re-action (DJ in the figure).

Now as DE is taken to be equal to AB, and as angles DEG and EDG are respectively equal to angles ABC and BAC,

$\therefore$ the triangle DEG = triangle ABC in every respect,

and EG = DF = BC = height of the plane,

also DG = DJ = AC = lenght of the plane.

Hence W : P : R :: $b : h : l$,

and $\frac{W}{P} = \frac{\text{base of the plane}}{\text{height of the plane}}$.

Or, on the 'principle of energy,'

when W moves from A to C, *i.e.*, through the *effective distance* BC, P moves through a distance AB; and the *equation of work* becomes

W $\times$ height of the plane = P $\times$ base of plane,

i.e., $\frac{W}{P} = \frac{\text{base of the plane}}{\text{height of the plane}}$.

Again W : R : : $b : l$,

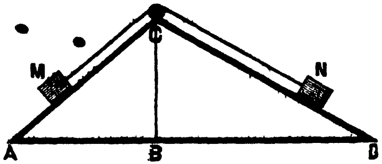
and as the re-action of the plane is equal though opposite in direction to the pressure on the plane,

$$\therefore \left\{\begin{matrix}\text{Pressure} \\ \text{on the plane}\end{matrix}\right\} : W :: \left\{\begin{matrix}\text{length} \\ \text{of} \\ \text{the plane}\end{matrix}\right\} : \left\{\begin{matrix}\text{base} \\ \text{of} \\ \text{the plane.}\end{matrix}\right\}$$

78. If two weights are attached to the extremities of a cord, which passes over a pulley at the vertex of a *double* inclined plane, then it is interesting to find the ratio between the weights which will produce equilibrium.

Let M and N be the two weights in equilibrium on the double inclined plane ACD.

Then it is evident that M is kept in position by the tension of the cord parallel to length AC, and N is similarly supported by the tension in the same cord along CD.



Therefore, for the weight M to remain in equilibrium on the inclined plane ABC, the power being parallel to the length of the plane, by article 76,

$$\frac{\text{Weight M}}{\text{Tension of the cord}} = \frac{\text{length of the plane}}{\text{height of the plane}} = \frac{AC}{CB};$$

and similarly for the weight N on the inclined plane CBD,

$$\frac{\text{Weight N}}{\text{Tension of the cord}} = \frac{CD}{CB}.$$

Compounding the two equations, we get

$$\frac{\text{Weight M}}{\text{Weight N}} = \frac{AC}{CD}.$$

Each of the **weights** is, therefore, **proportional to the length of plane on which it rests.**

EXAMPLES.

1. The length of an inclined plane is 18 feet, and the perpendicular height 6 feet; what power, acting parallel to the plane, will be required to sustain a weight of 1 cwt.?

The condition of equilibrium is $\frac{W}{P} = \frac{\text{length of the plane}}{\text{height of the plane}}$.

$\therefore$ By substitution

$$\frac{1}{P} = \frac{18}{6}$$

$$18P = 6$$

$$\therefore P = \tfrac{1}{3}\text{ cwt.}$$

2. The length of an inclined plane is 8 ft., the perpendicular height 2 ft., and the weight of the body is 12 lbs.; what power is required to prevent it from sliding down, supposing it acts parallel to the length of the plane? *Ans.* 3 lbs.

3. The lenght of an inclined plane is 120 ft., perpendicular height 20 ft., and the weight of the body 60 lbs.; find P. *Ans.* 10 lbs.

4. What load will be supported by a force equal to the weight of 1 cwt. on an inclined plane, the length of which is 100 ft., and height 5 ft., the force acting in the direction of the length of the plane? *Ans.* 20 cwts.

5. An inclined plane rises 28 in 100; what power, acting parallel to the plane, will support a weight of 1 ton?

The conditiou of equilibrium is

$$\frac{W}{P} = \frac{\text{lenght of the plane}}{\text{height of the plane}},$$

$$\text{and as } \frac{\text{lenght of the plane}}{\text{height of the plane}} = \frac{100}{28};$$

$\therefore$ by substitution

$$\frac{1}{P} = \frac{100}{28}$$

$$100\,P = 28$$

$$P = \frac{28}{100} = \frac{7}{25} \text{ of a ton} = 5\tfrac{3}{5} \text{ cwts.}$$

6. A railway train weighing 30 tons is supported upon an incline which rises 1 ft. in every 120 ft. by means of a rope parallel to the incline: find the strain brought to bear upon the rope. *Ans.* 5 cwts.

7. Upon an incline of 2 in 25 a railway train stops and is kept from sliding down by the friction of the rails: what is the amount of friction if the train weighs 125 tons? *Ans.* 10 tons.

8. An inclined plane rises 3 feet 6 in. for every 7 ft. of length: if $W = 100$ lbs. find P when it acts parallel to the length of the plane. *Ans.* 50 lbs.

9. A body weighing 7 lbs. is supported on a plane that rises 1 in 7 by a force that acts parallel to the plane. If 3 lbs. be added to the weight, what force will be required to support it? *Ans.* $1\tfrac{3}{7}$ lbs.

10. Three artillerymen drag a gun weighing 17 cwts. up a hill rising 2 in 17. Supposing the resistance to the wheels to be 16 lbs. per cwt. what pull parallel to the plane must each exert to move it?

Ans. $85\tfrac{1}{3}$ lbs. each.

11. When a certain inclned plane ABC, whose length is AB, is placed on AC as base, a power of 3 lbs. can support on it a weight of 5 lbs.: find the weight which the same power could support is the plane were placed on BC as base, so that AC is then the height of the plane. *Ans.* $3\frac{3}{4}$ lbs.

12. A mass of 10 lbs. is kept from sliding by a force in the direction of a smooth inclined plane rising 3 in 5. Find the force and the pressure on the plane. *Ans.*; 6 lbs.; 8 lbs.

13. Find what force is necessary to support a weight of 50 lbs. upon a plane inclined at 30° to the horizon, the force acting parallel to plane.

(See figure, art. 76.)

By Euclid, $\quad AC = 2\ BC$

$$\frac{AC}{BC} = \frac{2}{1}.$$

The condition of equilibrium is $\frac{W}{P} = \frac{\text{length of the plane}}{\text{height of the plane}}$.

$\therefore$ by substitution

$$\frac{50}{P} = \frac{2}{1}$$

$$2P = 50$$

$$\therefore P = 25 \text{ lbs.}$$

14. What weight could be supported, on a plane inclined at an angle of 30° to the horizon, by a man who was just strong enough to lift vertically a weight of 100 kilogrammes? *Ans.* 200 kilos.

15. A weight of 1 cwt. is supported on an inclined plane, whose inclination to the horizontal plane is 45o, by a string which passes over a pulley at the top of the plane and has a weight attached to it. Find the least possible value of the supporting weight. *Ans.* $\frac{1}{\sqrt{2}}$ cwt.

16. The inclinatfon of a plane is 30°, and a weight of 10 ozs. is supported on it by a string bearing a weight at its extremity, which passes over a smooth pulley at its summit: find the tension in the string.

Ans. 5 ozs.

17. What power acting parallel to the plane will raise a weight of 180 lbs. when the angle of inclination of the plane to the horizon is (*a*) 30°, (*b*) 45°, (*c*) 60°?

Ans. (*a*) 90 lbs., (*b*) $90\sqrt{2}$ lbs., (*c*) $90\sqrt{3}$ lbs.

18. A railway truck weighing 2 tons is supported on a smooth inclined plane, rising 1 in 200, by a rope fastened to stakes driven in the ground find the tension of the rope. *Ans.* $22\frac{2}{5}$ lbs.

19. A weight of 20 tons rests upon an inclined plane rising 1 in 100 and is prevented from sliding down by two ropes fastened to the weight and to a point in the plane: find the tension in each rope. *Ans.* 2 cwts.

20. A plane rises 3 in 8; what force parallel to it is required just to move a weight of 10 kilogrammes? *Ans.* 3·75 kilos.

21. A plane rises 1·25 in 12·5; what force parallel to it is required just to move a weight of 20 kilogrammes? *Ans.* 2 kilos.

22. The height of the plane is to its length as 1 to 100; what weight can be supported by a power of 12 lbs. acting along the plane? *Ans.* 1200 lbs.

23. With a force of 3 kilogrammes acting parallel to the plane what weight can be supported on a plane that rises 2 in 7? *Ans.* $10\frac{1}{2}$ kilos.

24. If the height of an inclined plane be 6 feet and its base 8 feet, what power, acting parallel to the plane, would be necessary just to support a weight of 10 maunds on the incline?

(See second figure, art. 75.)

By Euclid, Bk. I., Prop. 47,

$$AC = \sqrt{8^2+6^2} = \sqrt{100} = 10.$$

The condition of equilibrium is $\frac{W}{P} = \frac{\text{length of the plane}}{\text{height of the plane}}$.

∴ By substitution

$$\frac{10}{P} = \frac{10}{6}$$

$$10\,P = 60$$

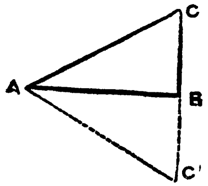
$$P = 6 \text{ maunds.}$$

25. Calculate the amount of force necessary to support a weight of 100 lbs. on an inclined plane, the height and base ot which are 3 ft. and 4 ft. respectively, on the supposition that the force is applied parallel to the slope of the plane. *Ans.* 60 lbs.

26. A plane rises 3 ft. vertically for every 5 ft. of length. Find the force, parallel to the plane, which supports a weight of 100 lbs. Find also the resistance of the plane. *Ans.* 60; 80 lbs.

27. A plane rises 3 in 5. Into what two parts must a weight of 40 lbs. be divided, so that there may be equilibrium with one part hanging over the top of the plane by a string fastened to the other part which rests on the plane? *Ans.* 25 and 15 lbs.

28. A power of 3 lbs. acting parallel to the plane is just enough to keep a weight of 6 lbs. on a smooth inclined plane. Find the inclination of the plane.



The condition of equilibrium is

$$\frac{W}{P} = \frac{\text{length of the plane}}{\text{height of the plane.}}$$

Let AC be the length of the plane and BC the height.

Then by substitution

$$\frac{6}{3} = \frac{AC}{BC},$$

$\therefore$ BC : AC :: 3 : 6

:: 1 : 2

Produce CB to BC′ and make BC′ = BC and join AC′;

then AC = CC′.

Again $AC^2 = AB^2 + BC^2$

and $AC'^2 = AB^2 + BC'^2$

$\therefore AC^2 = AC'^2$

and AC = AC′,

$\therefore$ triangle ACC′ is equilateral;

and angle CAC′ = 60°.

Now in the triangles ABC and ABC′, angles ABC and ABC′ are equal being right angles, and angles ACB and AC′B are also equal because they are each 90°

$\therefore$ angle CAB = angle C′AB

= ½ of angle CAC′

= ½ of 60° or 30°.

29. If a weight W be supported on an inclined plane by a force $\frac{W}{2}$ parallel to the plane, what is the inclination of the plane? *Ans.* 30°.

30. A weight of 20 lbs. is supported by a string fastened to a point in an inclined plane, and the string is only just strong enough to support a weight of 10 lbs., the inclination of the plane to the horizon being gradually increased : find when the string will break. *Ans.* After the inclination exceeds 30°.

31. A certain force, when acting along an inclined plane, can just support a weight of 10 lbs., and when acting on a wheel and axle, where

the diameter of the wheel is six times the diameter of the axle, it can support a weight of 30 lbs.; find the mechanical advantage and the inclination of the plane. *Ans.* 2; 30°.

32. A force of 15 lbs. when acting parallel to the plane can prevent a weight of 90 lbs. from sliding down the plane, which is perfectly smooth. Find the inclination of the plane to the horizon. *Ans.* 1 in 6.

33. A weight of 50 lbs. is supported by a string fastened to a point in an inclined plane and is only just strong enough to support a weight of 10 lbs. If the length of the plane be 10 ft., what will be the greatest height to which the plane can be tilted without the string breaking? *Ans.* 2 ft.

34. Calculate the amount of horizontal force necessary to support a weight of 50 lbs. on an inclined plane whose height and base are respectively 25 and 30 yards.

The conditon of equilibrium is $\frac{W}{P} = \frac{\text{base of the plane}}{\text{height of the plane}}$

∴ by substitution

$$\frac{50}{P} = \frac{30}{25}$$

$$6\ P = 250$$

$$\therefore P = 41\tfrac{2}{3} \text{ lbs.}$$

35. A plank rests upon a dray $3\frac{1}{2}$ ft. high and upon the ground at a distance of 7 ft. from the dray: find the force required to be applied horizontally, if the weight is 500 lbs. *Ans.* 250 lbs.

36. What weight can be supported on a plane by a horizontal force of 15 lbs. if the height is to the base of the plane as 3 to 4? *Ans.* 20 lbs.

37. A weight is kept upon an inclined plane by a horizontal force of 100 lbs., the height of the plane is 3 ft., the base is 5 ft.; find the weight. *Ans.* $166\frac{2}{3}$ lbs.

38. What weight can be supported on a plane by a horizontal force of 1 lb. if the height be to the base as 5 is to 8? *Ans.* $1\frac{3}{5}$ lbs.

39. What weight can be supported on a plane by a horizontal force of 10 ozs., if the ratio of the height to the base is 3 : 4? *Ans.* $13\frac{1}{3}$ ozs.

40. Find the pressure exerted on an inclined plane rising 16 in 80 by a weight of 560 lbs. resting on it and supported by a power acting horizontally.

(See figure, art. 77).

By Euclid, Bk. I., Prop. 47,

$$80^2 = 16^2 + \text{Base}^2$$

$$\therefore \text{Base} = \sqrt{80^2 - 16^2}$$

The condition of equilibrium is $\frac{W}{Pr} = \frac{\text{base of the plane}}{\text{length of the plane}}$.

$$\therefore \text{By substitution } \frac{560}{Pr} = \frac{\sqrt{80^2 - 16^2}}{80}$$

$$\sqrt{6144} \times Pr = 44800$$

$$Pr = \frac{44800}{\sqrt{6144}} \text{ lbs.}$$

41. The length of an inclined plane is 18 ft., and the perpendicular height 6 ft.; what power will be required to sustain a weight of 1 cwt.? *Ans.* $28\sqrt{2}$ lbs.

42. If the weight be 10 lbs. and the power 5 lbs., what is the pressurer on the inclined plane? *Ans.* $5\sqrt{3}$ lbs. if P acts along the plane
and $10\sqrt{5}$ lbs. if P acts horizontelly.

43. What power acting parallel to the base will raise a weight of $70\sqrt{3}$ lbs. when the angle of inclination of the plane to the horizon is (a) 30°, (b) 45°, (c) 60°. *Ans.* 70 lbs.; $70\sqrt{3}$ lbs.; 210 lbs.

44. What power acting horizontally will suffice to draw a weight of 96 lbs. up an inclined plane of which the length is 25 yds. and height 21 ft? Find also the pressure on the plane. *Ans.* 28 lbs.; 100 lbs.

45. If the pressure on an inclined plane be double the weight, what is the angle of the plane? *Ans.* 60°.

46. Find the inclination of a plane on which a horizontal force will support a weight equal to itself. *Ans.* 45°.

47. A body weighing 9 lbs. is an equilibrium upon an inclined plane under the action of a horizontal force of $3\sqrt{3}$ lbs., what is the inclination of the plane? *Ans.* 30°.

48. A force of 40 lbs. acting parallel to the base sustains a weight of 56 lbs. on an inclined plane whose base is 343 ft.; find the height of the the plane. *Ans.* 245 ft.

49. A weight of $10\sqrt{3}$ lbs. is supported by a horizontal force of 30 lbs. find the angle of the plane. *Ans.* 60°

50. Two inclined planes of the same height slope in opposite directions and two weights rest on each plane, connected with each other by a cord passing over a pulley at the common vertex. If the lengths of the planes are 5 ft., and 6 ft., find the relation of the weights that equilibrium may be possible.

The condition of quilibrium is $\frac{W}{W'} = \frac{L}{L'}$ (*vide* art. 78.)

By substitution, therefore, $\frac{W}{W'} = \frac{5}{6}$

The weight on the shorter incline must be to the weight on the longer incline as 5 : 6.

51. Two inclined planes of the same height, one of which is 8 ft. long, and the other 5 ft., are placed so as to slope in opposite directions and so that their summits may coincide. A weight of 20 lbs. rests on the shorter and is connected by a string passing over a pulley at the highest point of the planes with a weight resting upon the longer plane: find this weight, supposing there is equilibrium. *Ans.* 32 lbs.

52. Two inclined planes are placed back to back. They have the same height but their lengths are 4 : 5. Two weights rest upon the planes and are connected by a cord which passes over a pulley at the top of the planes. If the weight on the plane whose length is 4 ft. be 100 lbs. what will the other weigth be? *Ans.* 125 lbs.

53. On two inclined planes of equal height, two weights are repectively supported by means of a string passing over the common vertex and parallel to the planes. The length of one plane is double its height, and the length of the other is double its base. Show that the pressure on one plane is three times the pressure on the other.

54. AB, AC are two smooth planes inclined to the horizon at 60° and 30° respectively; a weight P on AB, and a weight Q on AC are connected by a string which passes over a pulley at A. If P and Q are in equilibrium, what is the ratio of their weights? *Ans.* 1 : $\sqrt{3}$.

55. The angle of a plane is 45°; what weight can be supported by a horizontal force of 2 ozs. and a force of 4 ozs. parallel to the plane, both acting together? *Ans.* 8·6 ozs. nearly.

56. Which will support the greater weight, a power acting horizontally, or the same power acting parallel to the plane?

In the first case $\frac{W}{P} = \frac{B}{H}$, $\therefore$ W $= \frac{B.P}{H}$;

In the 2nd case $\frac{W}{P} = \frac{L}{H}$, $\therefore$ W $= \frac{L.P}{H}$.

Now the denominators are the same but the numerator in the second case is greater than in the first; therefore the second arrangement is more advantageous.

57. The height, base, and length of an inclined plane are respectively 3, 4, and 5 ft. If the weight placed upon the plane be 50 lbs., find the reaction ot the plane, supposing the power to be applied (1) horizontally and (2) parallel to the length of the plane? *Ans.* 82½ lbs., 40 lbs.

58. A heavy body is just supported on a smooth inclined plane, the height of which is 3 ft. and length 5 ft., by the combined action of a horizontal force equal to 7 lbs. and a force acting along the plaue equal to 6 lbs. Determine the weight of the body. *Ans.* 19⅓ lbs.

59. If a weight of 160 lbs. be supported on an inclined plane by a power of 120 lbs., acting horizontally; what power, acting along the plane will support the same weight? *Ans.* 96 lbs.

50. If the weight supported by a force acting along the plane be double the weight supported by the same force acting horizontally, find the inclination of the plane. *Ans.* 60.

QUESTIONS.

1. What is the *general* direction of the reaction of a weight resting on a surface? When is it equal to the weight itself?

2. Define an *inclined plane*; also its *length*, *base* and *height*. What is the *inclination* of the plane or its *gradient*?

3. Why is the reaction less than the weight of a body when the body rests on an inclined surface? Hence deduce the forces which act on a body placed on an inclined plane, when the body is in equilibrium.

4. In what two different ways may the 'power' be applied on an inclined plane?

5. Find the relation between P, W and the Pressure on the plane (or the equal resistance of the plane) when the power is applied parallel to the plane.

6. Hence determine the 'mechanical advantage' to be obtained in the above case.

7. Find also the relation between P, W and Pressure on the plane when the power acts parallel to the base. What is the 'mechanical advantage' in this case?

CHAPTER VI.

The Wedge.

79. Def. The wedge is a machine composed of **two inclined planes placed base to base.**

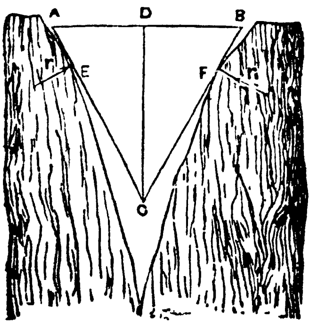
Unlike ordinary inclined planes, the wedge moves forward when power is applied, to it, hence it is frequently described as a **"moveable double inclined plane."**

By forcing such a body, for instance, below the bottom of an upright post, we can raise the post a few inches. Ships are raised in docks on the same principle by driving wedges placed underneath them. But the wedge is mainly used in splitting timber, stones, &c. Ploughs, hatchets, chisels, nails, swords and knives are all modifications of the wedge.

The side A B is called the **back** of the wedge and the line in which the two sides of the wedge meet is called the **edge.**

In the wedge the **force** is applied by sudden blows or impulses and **not** in a **continuous and steady** manner as in other machines.

80. Condition of equilibrium in an isosceles wedge.



Let ABC be the section of an isosceles wedge introduced into a piece of wood and let the points of contact E and F be similarly situated. Let a power P act at D, the centre of the back of the wedge, and suppose the wedge has been driven into the wood through the distance CD.

Then we may consider *the wedge to be a pair of inclined planes placed base to base*, viz., ADC and BDC. Let the resistances offered by the wood at E and F be each equal to r and let the power P be equally divided between the two inclined planes, each half being equal to p. Then as the power will be acting along the bases of the inclined planes, by art. 77, for the half wedge ADC, for equilibrium,

$$p : r :: h : l$$

$$\text{or} \quad p = r.\frac{h}{l} = r.\frac{\text{AD}}{\text{AC}}$$

$$= r.\frac{\text{half the back of the wedge}}{\text{length of the wedge.}}.$$

Similarly for the other half wedge BDC, for equilibrium

$$p = r.\frac{\text{BD}}{\text{BC}},$$

$$= r.\frac{\text{half the back of the wedge}}{\text{length of the back}}.$$

Adding the two equations and putting P for $2p$ and R for $2r$, we get

$$\text{P} = \text{R} \times \frac{\text{half the back of the wedge}}{\text{length of the wedge}}.$$

$$\text{or} \quad \frac{\text{P}}{\text{R}} = \frac{\text{half the back of the wedge}}{\text{length of the wedge}}.$$

Therefore, the **power** is to the total **resistance** overcome, **as half the back of the wedge is to the length** of one of the sides.

The **machanical advantage** is measured by $\frac{\text{R}}{\text{P}} = \frac{\text{AC}}{\frac{1}{2}\text{AB}}$, therefore, by a making AB small, *i.e.*, by lessening the size

of the back and also by increasing AC, in other words by making the wedge thin, the mechanical advantage is increased, because the value of $\frac{AC}{\frac{1}{2}AB}$ is increased thereby.

EXAMPLES.

1. If an isosceles wedge be a decimetre long in the sides and a centimetre wide at the back, find the total force exerted on the timber by a force of 200 kilogrammes.

The condition of equilibrium is

$$\frac{R}{P} = \frac{\text{length of the side}}{\text{half of the back}}.$$

$$\therefore \text{ By substitution } \frac{R}{200} = \frac{10}{0{\cdot}5}$$

$$\text{and } 0{\cdot}5\ R = 2000$$

$$R = 4000 \text{ kilogrommes.}$$

2. If a wedge be 12 ins. long and 1 in. wide at the base, find the force exerted on the timber by a pressure of 250 lbs. on the end of the wedge. *Ans.* 6,000 lbs.

3. In a wedge, the slant side is 12 ins., the thickness 3 ins., and the pressure produced by a blow from the hammer is 750 lbs., determine the resistance overcome one each side of the wedge. *Ans.* 3,000 lbs.

4. If one of the slant sides of a wedge be 15 ins long and the thickness of the wedge is 3 ins., find the force exerted on a block of timber by a pressure of 100 lbs. applied on the end of the wedge. *Ans.* 1000 lbs.

5. A wedge has its slant side 12 ins. long, and its thickness 1 in.: determine the resistance when the power is equal to 100 lbs. *Ans.* 2400 lbs.

6. A wedge is right-angled and isoceles; the pressure exerted upon the side opposite to the right angle is 100 lbs.: find by construction the reaction upon each of the other faces. *Ans.* 70·7 lbs.

7. A wedge is right-angled and isosceles, and a force of 50 lbs. acts opposite to the right-angle: determine the two forces. *Ans.* $25\sqrt{2}$ lbs.

8. Compare the resistances offered to the two wedges when a power of 75 lbs. is applied to each in splitting a log of timber: 1st wedge—length 15 ins., thickness 3 ins.; and 2nd wedge—length 20 ins., thickness $3\frac{1}{2}$ ins. *Ans.* 750 lbs.; $850\frac{1}{7}$ lbs.

9. Three wedges are employed in splitting stones in a quarry: you are required to state which of the three is most effective, the pressure exerted in each case being 200 lbs.; 1st wedge—length 7 ins., thickness 2 ins.; 2nd wedge—length 8 ins., thickness 3 ins.; 3rd wedge—length 9 ins., thickness $3\frac{1}{2}$ ins. *Ans.* The first.

10. A wedge is in the form of an equilateral triangle and the two side forces are 40 lbs. each; find the third force.

Total resistance (R) is equal to 80 lbs. and as the back of the wedge is of the same size as the length,

$$\therefore \frac{R}{P} = \frac{\text{length}}{\frac{1}{2}\text{ length}},$$

and by substitution $\frac{80}{P} = \frac{1}{\frac{1}{2}}$, or $P = 40$ lbs.

11. A wedge is in the form of an equilateral triangle and the two re-actions are 50 lbs. each: find the pressure exerted upon the third side. *Ans.* 50 lbs.

12. The resistance offered by a wedge, whose length is 20 inches and breadth 3 inches, is 1600 lbs. Find the pressure on the back. *Ans.* 120 lbs.

13. Find, by mechanical construction, the angle of an isosceles wedge when the pressure on the end opposite to this angle is equal to the re-action on one of the slant sides. *Ans.* 60°.

14. Find the vertical angle of an isosceles wedge when the pressure on the face opposite this angle is equal to half the sum of the two resistances. *Ans.* 60°.

15. Determine the shape of a wedge when the pressure on the back is equal to the re-action on one of the slant sides. *Ans.* Equilateral.

16. A force of 200 lbs. overcomes a resistance of $200\sqrt{2}$ lbs., find the vertical angle of the wedge. *Ans.* 90°.

17. The pressure being 50 lbs., the reaction at each end 500 lbs., and the slant side 11 inches, what is the thickness of the wedge? *Ans.* $1\frac{1}{10}$ in.

QUESTIONS.

1. Define a *wedge*; which is its *back*? What is the *edge* of a wedge?

2. State for what particular kinds of work the wedge is specially adapted. Mention some of the uses to which it is put in practice.

3. Find the 'mechanical advantage' in an isosceles wedge.

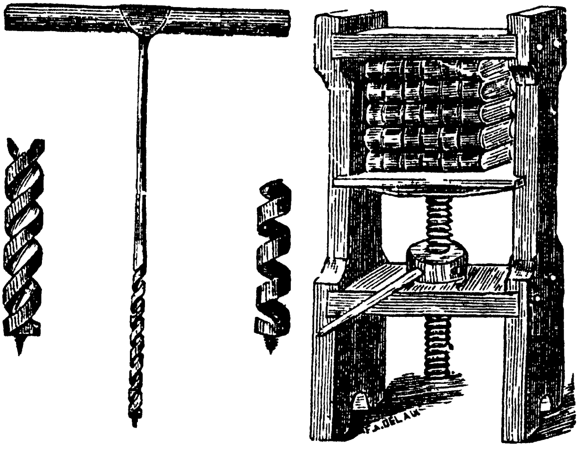
4. With the same length of the wedge, why is a thin wedge to be preferred to a broad one?

5. Explain why a ship is usually given such "fine lines"?

CHAPTER VII.

The Screw.

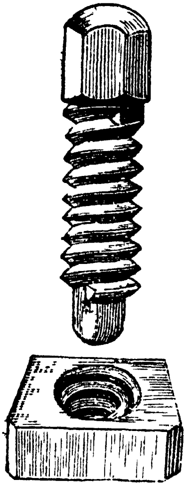
81 **Def.** The screw consists of **cylinder** (either solid or hollow) **with a uniform projecting thread** traced round its surface and inclined **at a constant angle** to lines parallel to the axis of the cylinder.



The screw-auger, the copying-press, the book-binder's press, and the corkscrew are some of the more familiar instances of the application of the screw to the purposes of evrey-day life.

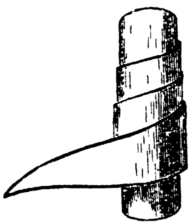
82. Companion screws; right-handed and left-handed screws.

Def. A solid and a hollow screw, fitting into one another, form **companion screws**. The former is called the **bolt** or **male**-screw, and the latter the **nut** or **female**-screw.



When in moving forward or driving in a screw if it is necessary to twist it in the direction of the hands of a watch, *i. e., from right to left*, the screw is called **right-handed**, but if the thread be cut in such a manner as to require the screw to be turned in the opposite direction while driving it in, it is called **left-handed**.

83. Connection between a screw and an inclined plane.—The screw is in reality an **inclined plane wrapped round a cylinder.** If a slip of paper cut in the form of an inclined plane be taken and its '*height*' fastened to a cylinder lengthwise and the paper wrapped tightly round the cylinder (as shown in the figure), then the '*length*' *of the inclined plane will form the thread* of the screw.



It will be observed that the spiral is inclined everywhere at the same angle to the horizon.

Def. The angle on which depends the slope of the thread is called the **inclination** of the thread.

The successive parts of the thread are parallel to each

other ; therefore if we take a point in one thread and draw a vertical line to the thread directly above, the vertical distance remains constant.

Def. The vertical distance between the two consecutive threads is called the **pitch** of the screw.

84. Relation between the power and weight in the screw when there is equilibrium.

When the screw is used for mechanical purposes, the **power** is **applied** in a direction **perpendicular to the axis** of the cylinder, *i. e.*, parallel to the base of the inclined plane forming the screw (*vide* figures above) ; and the **weight** is the *resistance overcome by the thread* of the screw, which is **along the length** of the inclined plane forming the screw.

The problem is, therefore, one in which P is applied parallel to the base of an inclined plane with W resting on the length of the plane (*vide* Art. 77). The condition of equilibrium, therefore, is

$$\frac{W}{P}=\frac{\text{base of the inclined plane}}{\text{height of the inlined plane}};$$

but the base of the plane described in one turn of the screw is equal to the circumference through which the power moves, and the height of the inclined plane through the same range is equal to the distance between two consecutive threads ;

$$\therefore\ \frac{\mathbf{W}}{\mathbf{P}}=\frac{\text{circumference of the circle described by P}}{\text{distance between two consecutive threads}}.$$

Or, on the 'principle of energy,' P × the circumference of the circle through which P moves = W × the distance through which it is raised ; and if P acts, as is usually the case, at the end of horizontal lever, then at every one

complete turn or revolution of the lever the screw advances through the distance between two consecutive threads and the *equation of works* take the form—

P $\times$ circumference of power-handle $=$ W $\times$ distance between two consecutive threads ;

$$\text{hence, } \frac{W}{P} = \frac{\text{circumference of power-handle}}{\text{distance between two consecutive threads}}.$$

From the expression for $\frac{W}{P}$ just obtained, it will be seen that the 'mechanical advantage' in a screw *increases with the closeness* of the threads, because reducing the distance between consecutive threads increases the value of $\frac{W}{P}$.

EXAMPLES.

1. If the screw move forward $\frac{1}{2}$ in. whilst a power of 10 lbs. describes a revolution of 10 ft., what pressure is produced ? *Ans.* 2400 lbs.

$$\frac{W}{P} = \frac{\text{Circumference of power-handle}}{\text{pitch}}.$$

$$\frac{W}{10} = \frac{10 \times 12}{\frac{1}{2}}$$

$$\therefore W = 2400 \text{ lbs. } \textit{Ans.}$$

2. If on a bolt whose length is 1 ft. there are 100 threads, what is the pitch of the screw? *Ans.* $\frac{3}{25}$th of an inch.

3. A screw has 8 threads to the inch, what distance will a weight be lifted in three turns of the lever? *Ans.* $\frac{3}{8}$ ths. of an inch.

4. If a power of 1 kilogramme describe a revolution of 2 metres, whilst the screw moves through 2 centimetres, what pressure will be produced ? *Ans.* 100 kilos.

5. If the circumference of a screw be 10 ins., what force must be applied to overcome a resistance of 30 lbs., the distance between the threads being $\frac{1}{4}$ inch ? *Ans.* $\frac{3}{4}$ lb.

6. The circumference described by the power is 4 ft., and the distance between the threads is $\frac{1}{3}$ in. What power is required to produced a pressure of 1 ton? *Ans.* $15\frac{5}{9}$ lbs.

7. A screw having a head 12 ins. in circumference is so formed that it advanced a quarter of an inch at each revolution: find what force must be applied at the circumference of the head that the screw may produce a pressure of 96 lbs. *Ans.* 2 lbs.

8 If the circumference of the circle described by the extremity of the power-arm be 12 ft., and the pitch of thes crew be $\frac{1}{4}$ inch, what pressure must be applied to support a weight of 5 cwts. *Ans.* $\frac{35}{36}$th of a lb.

9. If the circumference of a screw be 12 ins., what force must be applied to support a weight of 50 lbs., the distance between the threads being $\frac{1}{5}$th of an inch. *Ans.* $\frac{5}{6}$th of a pound.

10. The circumference of the circle described by the extremity of the power-arm being 5 ft. and the pitch of the screw being $\frac{1}{10}$th of an inch, what power is required to support a weight of 2 tons? *Ans.* $7\frac{7}{15}$ lbs.

11. The circumference of a screw is 1 ft. and the distance between two threads $\frac{1}{8}$th of an inch. What power must be applied to overcome a resistance of 50 lbs? *Ans.* $\frac{25}{48}$th of a lb.

12. What power will be required to raise 1,000 lbs. upon a screw which has 10 threads to the inch, when the power passes over a distance of 3 ft. in one revolution? *Ans.* $2\frac{7}{9}$ lbs.

13. A screw, one foot in diameter, having one inch between the threads, is turned by a capstan with 6 bars, each 4 feet long: if the power applied to each bar be 56 lbs., what is the pressure at the end of the screw?

For the condition of equilibrium, *vide* Art. 84.

As the screw is 1 foot in diameter and the capstan-bars 4 ft. long, P is applied at a distance of $4\frac{1}{2}$ ft. from the centre, and the circumference of the circle described by $P = \frac{22}{7} \times 2 \times 4\frac{1}{2} \times 12$ or $\frac{2376}{7}$ inches, and $P = 56 \times 6 = 336$ lbs.

∴ By substitution

$$\frac{W}{336} = \frac{\frac{2376}{7}}{1}$$

$$\therefore \quad W = \frac{2376}{7} \times 336 = 2376 \times 48 = 114{,}048 \text{ lbs.}$$

14. In a screw which has seven threads to the inch, find the pressure that can be produced by a force of 6 lbs. applied at the circumferemce, the radius of the cylinder being 1 inch. *Ans.* 264 lbs.

15. A screw is made to revolve by a force of 2 kilogrammes applied at the end of a lever 1 metre long; if the distance between the threads be 1 centimetre, what pressure can be produced? *Ans.* $1257\frac{1}{7}$ kilos.

16. In a common screw the interval between the threads is $\frac{1}{11}$ of an inch and the radius of the circle described by the power 2 ft. What resistance will a power of 28 lbs. sustain? *Ans.* 46,464 lbs.

17. In a screw which has 15 threads to the inch, what weight can be supported by a power of 10 lbs., applied at the circumference, the diameter of the screw being 3 ins.? *Ans.* $1414\frac{2}{7}$ lbs.

18. A screw is made to revolve by a force of 4 lbs. applied at an end of a lever $3\frac{1}{2}$ ft. long. If the distance between the threads be $\frac{1}{4}$ in., what pressure is produced? *Ans.* 4,224 lbs.

19. A screw whose pitch is $\frac{1}{4}$ in. is turned by means of a levor 4 ft. long: find the power which will raise 15 cwts. *Ans.* $1\frac{69}{176}$ lbs.

20. In a common press the screw fs 12 ins. long, and the thread goes round it 20 times, tne length of the power-arm is 2 ft., what power will sustain a weight of 352 lbs? *Ans.* $1\frac{2}{5}$ lb.

21. In a common press the screw is 12 ins. long, and the thread goes twenty times round it, and the length of the lever measured from the axis is 2 feet; what power will produce a pressure of 335 lbs.? *Ans.* $1\frac{117}{352}$ lbs.

22. A screw is employed to move a body horizontally. The thread of the screw is $\frac{1}{2}$ in. pitch, the lever is 5 ft. long, and the friction to be overcome is 100 lbs. What power will be required to do it, and how far will the body move for each complete revolution of the screw?

Ans. $\frac{35}{264}$ th of a lb.; $\frac{1}{2}$ in.

23. What is the ratio between the power and the weight in a screw which has eight threads to the inch. and is moved by a power acting perpendicularly to an arm at a distance of 2 feet from the centre?

The circumference of the circle described by P

$$= \frac{22}{7} \times 2r = \frac{22}{7} \times 2 \times 24 \text{ or } \frac{1056}{7} \text{ inches,}$$

and the distance between the threads $= 1 \div 8$ or $\frac{1}{8}$ inch.

Condition of equilibrium :—*vide* Art. 84.

∴ By substitution

$$\frac{W}{P} = \frac{\frac{1056}{7}}{\frac{1}{8}} = \frac{8448}{7}$$

$$\therefore \ P : W :: 7 : 8448.$$

24. Find what weight can be raised by means of a screw, with threads 5 millimetres distant by a force of one kilogramme applied to the end of a lever 5 decimetres long. (Ratio of circumference to diameter of a circle, 3 : 1.)

5 decimeters = 500 millimetres.

Therefore, the *diameter* of the circle described by P = 1,000 millimetres and circumference by $P = 3 \times d = 3 \times 1{,}000 = 3{,}000$ millimetres.

Condition of equilibrium :—*vide* Art. 84.

∴ By substitution

$$\frac{W}{1} = \frac{3000}{5}$$

$$W = 600 \text{ kilogrammes.}$$

25. What is the mechanical advantage on a screw, when the diameter is $1\frac{3}{4}$ in. and the distance between the threads $\frac{1}{4}$ inch? (The circumference to be taken as $\frac{22}{7}$ th of the diameter.

The circumference of the circle described by P

$= \frac{22}{7} \times d = \frac{22}{7} \times \frac{7}{4}$ or $\frac{11}{2}$ inches and the distance between the threads is $\frac{1}{4}$ inch.

Condition of equilibrium :—*vide* Art. 84.

∴ By substitution

$$\frac{W}{P} = \frac{\frac{11}{2}}{\frac{1}{4}} = 22.$$

26. The diameter of a screw is 2 ins.: and the distance between the threads is $\frac{1}{8}$th of an inch: find the mechanical advantage. *Ans.* $\frac{W}{P} = 50\frac{2}{7}$.

27. Calculate the mechanical advantage of a screw, the diameter of which is 6 inches and the distance between its threads $\frac{1}{2}$ inch.

Ans. $\frac{W}{P} = \frac{264}{7}$.

28. Suppose the circumference of the circle described by the power-arm is 100 inches, and the pitch of the screw to be $\frac{1}{4}$ in., what is the mechanical advantage? *Ans.* 400.

29. The diameter of a screw is 2 ins., and the distance between the threads is $\frac{1}{4}$ in., what is the mechanical advantage? *Ans.* $25\frac{1}{7}$ nearly.

30. The angle of a screw is 30°, and the length of the power-arm is a times the radius of the cylinder; find the mechanical advantage. *Ans.* $a\sqrt{3}$.

31. The angle of a screw is 30°, compare the pitch with the length of the thread passed over in one revolution of the lever. *Ans.* 1 : 2.

32. If the extremity of the power-arm describes a circle of 10 ft. and a force of 1 lb. supports 1 ton, what is the distance between the threads of the screw?

$$\frac{W}{P} = \frac{\text{cir. of power-arm}}{\text{pitch}}$$

$$\frac{112 \times 20}{1} = \frac{10\,\text{ft.}}{\text{pitch}}$$

$$\therefore \text{pitch} = \frac{10}{112 \times 20}\,\text{ft.} = \frac{3}{56}\,\text{in.} \qquad \textit{Ans.}\ \frac{3}{56}\,\text{in.}$$

33. Find the length of the lever of a book-binder's press in which a force of 1 lb. will support a pressure of 2 cwts. produced by the books, the distance between two contiguous threads being $\frac{1}{11}$ths. of an inch.

Ans. 28 inches.

34. A screw is used to produce a pressure of 500 lbs. by means of a power of 1 lb. acting at the end of a lever 5ft. long: find the pitch of the screw. *Ans.* 0·75 inches nearly.

35. How many turns will be made upon a screw formed upon a cylinder whose length is 14 ins. and circumference 7 ins., when a power of 3 lbs. raises a weight of 30 lbs.? *Ans.* 20.

36. What is the distance between the two adjacent threads of a screw when a power of 2 lbs. sustains a weight of 96 lbs., the distance between the centre of the head of the screw and the point of application of the power being 2 ft.? *Ans.* $3\frac{1}{7}$ in.

QUESTIONS.

1. Describe the *screw*. What is its *pitch*?

2. What are *companion-screws*? Describe the *bolt* and the *nut*. Distinguish between a *male-screw* and a *female-screw*.

3. Name and describe some of the more familiar applications of the screw.

4. The screw is frequently described as 'an inclined plane wrapped round a cylinder.' Explain the statement.

5. Obtain the value of $\frac{W}{P}$ in a screw and show that a screw with close threads has a high mechanical advantage.

SUMMARY OF RESULTS.

Force implies an agent which sets a body in motion or attempts to put it in motion, or brings a body to rest or makes an attempt to stop it. (§§ 1, 2, 3.)

Matter has *Mass* and inertia. (§ 5.)

A straight line can represent a force in all particulars, *viz.*, (1) point of application, (2) direction, and (3) magnitude. (§ 7.)

Gravity, pressure and **tension** are the chief static forces affecting matter. (§§ 6, 8, 9.)

Weight is the effect of gravity on the mass of a body. (§ 6.)

Opposite forces have opposite signs. (§ 10.)

The algebraical sum of **forces in equilibrium** all acting in a straight line is zero. (§ 10.)

Adding equal forces to each side of a system of forces does not change the final result. This is the **principle of superposition of forces.** (§ 12.)

The **principle of transmissibility** asserts that the effect of a force on a body is not altered by transferring it to any other point, if the original line of action is not changed and if the point of application is kept in rigid contact with the body. (§ 13.)

If **three forces**, not acting in a line, keep a body **in equilibrium**, two of them are greater than the third and these three forces must meet in a point. (§ 14.)

The **parallelogram of forces.**—If two forces acting at an angle on a point be represented in magnitude and direction by the adjacent sides of a parallelogram, the resultant will be represented in magnitude and direction by the diagonal of the parallelogram passing through this point. (§ 16.)

Half the parallelogram is sufficient to represent the two forces and their resultant, but only as regards their magnitude and direction and not as to their point of application. (§ 18.)

By the 'parallelogram of forces,' either the **composition** or the **resolution** of forces my be effected. (§§ 17, 19.)

The **magnitude of the resultant** varies inversely as the angle between the forces and the direction of the resultant is inclined towards the greater force. (§ 22.)

Magnitude of the resultant of any **two forces** is given by the equation $R^2 = P^2 + Q^2 + 2\,P.\,Q.\,\sqrt{x}$.

For 0° $R = P + Q$.

„ 30° $R^2 = P^2 + Q^2 + P.\,Q.\,\sqrt{3}$

„ 45° $R^2 = P^2 + Q^2 + P.\,Q.\,\sqrt{2}$

„ 60° $R^2 = P^2 + Q^2 + P.\,Q.\,\sqrt{1}$

„ 90° $R^2 = P^2 + Q^2 + P.\,Q.\,\sqrt{0}$

„ 120° $R^2 = P^2 + Q^2 - P.\,Q.\,\sqrt{1}$

„ 135° $R^2 = P^2 + Q^2 - P.\,Q.\,\sqrt{2}$

„ 150° $R^2 = P^2 + Q^2 - P.\,Q.\,\sqrt{3}$

„ 180° $R = P \backsim Q$.

(§§ 23, 24.)

The **resultant of two like parallel forces**, P and Q, acting at A and B has its magnitude = P + Q, its direction parallel to those of P and Q and its point of application is such that P × CA = Q × CB. (§ 27.)

The **resultant of two unlike parallel** forces, P and Q, is such that its magnitude = P ∽ Q. its direction is parallel to those of P and Q) and its point of application is definite, *viz.*, at B produced in AC on the side of the greater force. (If Q be greater than P, then P × AB = Q × CB.)
(§ 28.)

Two unlike parallel forces when *equal* form a **couple**. (§ 29.)

Three forces in equilibrium must either be parallel or intersect at a point. (§ 30.)

The **resultant** of any number of **parallel** forces is the algebraical sum of the forces. Its point of application is called the **centre** of the parallel forces. (§ 31.)

The **moment** of a force P about a point D = P × perpendicular distance of P from D. (§ 32.)

The **moment** of the **resultant** of a number of **parallel** forces about any point is equal to the sum of the moments of its components. (§ 33.)

Parallel forces are in **equilibrium** when the algebraical sum of their moments is zero. (§ 34.)

The **centre of gravity** of a body is that point through which the resultant force due to the earth's attraction passes. (§ 35.)

A body can have only *one* centre of gravity. (§ 36.)

When a body is supported at a point, the point of support of suspension and the C. G. lie in the same vertical line. (§ 37.)

The **centre of gravity**

(*a*) of a *thin* uniform *rod* is its middle point,

(*b*) " " " *ring* is the centre of the ring,

(*c*) " " " *disc* is the centre of the disc,

(*d*) " " " *parallelogram* is the intersection of its diagonals,

(*e*) of a homogeneous *sphere* is its centre,

(*f*) " a *right cylinder* is the middle point of its axis,

(*g*) " a *rectangular parallelopiped* is at the intersection of its diagonals. (§ 39.)

A body will **stand or fall** according as the vertical line through the C. G. falls within or without the base on which the body rests. (§ 40)

The three **states of equilibrium** are stable, unstable, and neutral. (§ 41.)

The measure of **work** is $W \times d$. (§§ 42, 43.)

Energy can neither be *created* nor *destroyed*. (§ 45.)

The **equation of work** is $P \times d = W \times d'$. (§ 45.)

The **simple machines** are six, *viz.*, the lever, the wheel and axle, the pulley, the inclined plane, the wedge and the screw; these may all be arranged under three separate heads. (§ 47.)

The ratio $\frac{W}{P}$ in a machine is called the **modulus** of the machine. (§ 48.)

When $\frac{W}{P}$ is **greater than unity**, the machine is said to work at a **mechanical advantage**; when less, at a *disadvantage*. (§ 48.)

There is a physical **limit to mechanical advantage**. (§ 48.)

There are **three kinds of levers**. (§ 49.)

In a lever of the *first kind*, *fulcrum* is in the middle,

" " the *second kind*, *weight* " " "

" " the *third kind*, *power* " " " (§ 50, 51, 52.)

In all the three systems,

moment of P = moment of W round the fulcrum, (§ 53.)

or the work done by P = the work done by W. (§ 55.)

In a lever of the *first kind*, $\frac{W}{P}$ is > = or < 1,

the *second kind*, $\frac{W}{P}$ is always > 1,

the *third kind*, $\frac{W}{P}$ is always < 1. (§ 53.)

In a **compound** lever, the product of P into its several power-arms is equal to the product of W into its several weight-arms. (§ 56.)

The **balance** is a lever of the first kind with equal arms. (§ 57.)

The **requisites** of a good balance are that it should be (1) *true*, (2) *sensitive*, and (3) *stable*. (§ 58.)

If **true**, the beam remains horizontal both when the balance is empty and when it is equally loaded. For this the *conditions* are (*a*) equal arms, and (*b*) scale-pans of equal weight. (§ 59.)

It is **sensitive**, when it is sensibly deflected by a slight excess in the load on one side. This is accomplished (1) by reducing the weight of the beam, (2) by bringing the C.G. near the fulcrum, and (3) by increasing the length of the arms. (§ 60.)

It is **stable**, when it returns readily to its position of equilibrium after being displaced; for this the C.G. must be always below the fulcrum. (§ 61.)

To **increase stability** the centre of gravity must be lowered and the beam made heavy. (§ 61.)

True weight could be ascertained by a balance with unequal arm by taking a mean proportional between the two false weights. $W = \sqrt{PP_1}$. (§ 62.)

The **wheel and axle** is a modification of the lever. (§ 65.)

Its **condition of equilibrium** is $\frac{W}{P} = \frac{\text{radius of the wheel}}{\text{radius of the axle}}$. (§ 65.)

In **the compound wheel and axle**,

$$\frac{W}{P} = 2 \times \frac{\text{length of the handle}}{\text{difference between the radii of the two axles.}} \quad (\S\ 68.)$$

In the **fixed pulley**, P = W. (§ 71.)

In the **single moveable** pulley,

$\frac{W}{P} = 2$, tension = $P = \frac{1}{2}$ W, and pressure on the beam = $\frac{1}{2}$ W. (§ 72.)

If the pulley be heavy then $P = \frac{W+w}{2}$. (§72.)

When the cords are inclined $\frac{W}{P} < 2$. (§72.)

There are **three systems** of pulleys. (§73.)

In the first system of PULLEYS, $W = 2^n P$, or $P = \frac{W}{2^n}$;

„ second „ „ „ $W = nP$, or $P = \frac{W}{n}$;

„ third „ „ „ $W = (2^n - 1)P$, or $P = \frac{W}{2^n - 1}$.

With heavy pulleys,

in the first system, $P = \frac{W-w}{2^n} + w$;

„ second „ $P = \frac{W+w}{n}$;

„ third „ $P = \frac{W - (2^n - n - 1)w}{2^n - 1}$. (§ 37).

In the **inclined plane** there are three forces in equilibrium, *viz.*, the weight, the power, and the re-action of the plane. (§ 75.)

When the **power** acts **parallel to the plane**, $\frac{W}{P} = \frac{\text{length of the plane}}{\text{height of the plane}}$

and $\frac{W}{\text{Pressure in the plane}} = \frac{\text{length of the plane.}}{\text{base of the plane}}$ (§ 76.)

When the **power** acts **parallel to the base**, $\frac{W}{P} = \frac{\text{base of plane}}{\text{height of the plane}}$

and $\frac{W}{\text{Pressure of the plane}} = \frac{\text{base of the plane}}{\text{length of the plane}}$. (§ 77.)

The **wedge** is a double inclined plane. (§ 79.)

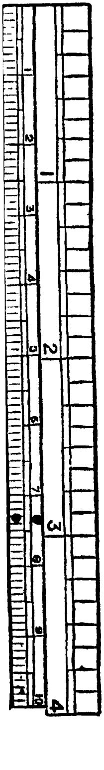
Its **condition of equilibrium** is $\frac{P}{R} = \frac{\text{half the back of the wedge}}{\text{length of the wedge}}$.

(§ 80.)

In the **screw** $\frac{W}{P} = \frac{\text{circumference of power handle}}{\text{distance between any two consecutive threads}}$ (§ 84.)

APPENDIX A.

The Metric System of Weights and Measures.



The **fundamental unit** on this system is the unit of **length**, called the **metre**, which is taken to be equal to **one-forty-millionth part of a meridian circle**.

The system being a **decimal system** all the *derivatives* of the metre are *multiples and submultiples of ten.*

Thus—

$$\text{milli-metre} = \frac{1}{1000}\ \text{metre.}$$

$$\text{centi-metre} = \frac{1}{100}\ \text{metre.}$$

$$\text{deci-metre} = \frac{1}{10}\ \text{metre.}$$

metre

deca-metre = 10 metres.
hecto-metre = 100 metres.
kilo-metre = 1000 metres.

One metre = 39·37 inches or 1·1 yard nearly. Hence one-tenth metre or ten centimetres are equal to 3·937 inches or 4 inches approximately.

The **unit of square** or superficial **measure** may be taken to be a square, each side of which is a metre and called a *square-metre.* The **arc** is a square whose sides are ten metres each.

The **unit of capacity or volume** is a cube, each side of which is one decimetre or ten centimetres; it is called a **litre.** As each side of the litre is ten centimetres long, there are **1000 cubic centimetres in one litre.**

Both the *are* and the *litre* may have their multiples and sub-multiples arranged as in the case of the metre, but they are seldom employed in scientific works.

As 1 metre = 39·37 inches,
0·1 metre
or
1 decimetre = 3·937 inches ;
$(1\ \text{decimetre})^3 = (3{\cdot}937\ \text{inches})^3$;
i.e., 1 litre = 61·03 cub. inches.

The unit of weight is the weight of one cubic centimetre of water at 4 C and is called a gramme.

$$\text{Milli-gramme} = \frac{1}{1000} \text{ gramme.}$$

$$\text{Centi-gramme} = \frac{1}{100} \text{ gramme.}$$

$$\text{Deca-gramme} = \frac{1}{10} \text{ gramme.}$$

Gramme

Deci-gramme = 10 grammes.

Hecto-gramme = 100 grammes.

Kilo-gramme = 1000 grammes.

As a litre contains 1000 cubic centimetres, a litre of water at 4° C. weighs 1000 grammes or one kilo-gramme.

One gramme = 15 432 grains nearly,

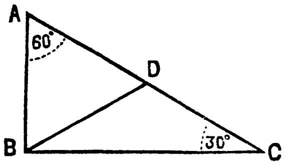
One kilo-gramme = 15,432 grains

or 2 205 lbs. avoir. nearly.

APPENDIX. B.

1. In a right-angled triangle, the side opposite the angle of 30° is half the hypotenuse.

Let ABC be a right-angled triangle, the angles at A, B and C being respectively 60°, 90° and 30°; then AB, the side opposite the angle of 30° shall be equal to $\frac{1}{2}$ AC.



Make angle CBD = angle DCB, (Eu., I., 23.) then in triangle BDC, BD = CD, (Eu., I., 6.)

Again, angle ABD = 90° — 30° = 60°,
and angle BAD = 60°,
therefore angle ABD = 60° (Eu. 1., 32.)

and the triangle ABD is equiangular and hence equilateral; (Eu. I., 6.)

$$\text{therefore, } AB = AD = BD;$$
$$\text{but } BD = CD,$$
$$\text{therefore, } AD = CD,$$
$$= \frac{1}{2} AC,$$
$$\text{hence also } AB = \frac{1}{2} AC;$$

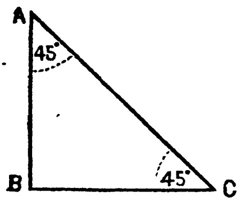
i.e., the side opposite the angle of 30° is equal to half the hypotenuse.
Q.E.D.

2. In a right-angled triangle the side opposite the angle of 60° is equal to half the hypotenuse multiplied by $\sqrt{3}$.

$$AC^2 = AB^2 + BC^2 \qquad \text{(Eu. I., 47.)}$$
$$\text{therefore, } BC^2 = AC - AB^2,$$
$$\text{but } AB = \frac{1}{2} AC,$$
$$\text{therefore, } BC^2 = AC^2 - (\tfrac{1}{4} AC^2),$$
$$= \frac{3}{4} AC^2;$$
$$\text{and } BC = \sqrt{\tfrac{3}{4} AC^2}$$
$$= AC . \frac{\sqrt{3}}{2}$$
$$= \frac{AC}{2} . \sqrt{3};$$

i.e., the side opposite the angle of 60° is equal to half the hypotenuse multiplied by $\sqrt{3}$.

3. **In right-angled triangle, the side opposite the angle of 45° is equal to the hypotenuse divided by $\sqrt{2}$**



Let ABC be a right-angled triangle having angles at A and C equal to 45°; then shall the sides AB and BC be each equal to $\frac{AC}{\sqrt{2}}$.

Because angle BAC = angle ACB,

therefore AB = CB,

Again $AC^2 = AB^2 + BC^2$ (Eu. I. 47)

$= 2\ AB^2$,

therefore, $AB^2 = \frac{AC^2}{2}$,

and $AB = \frac{AC}{\sqrt{2}}$;

Again AB = CB,

therefore also $CB = \frac{AC}{\sqrt{2}}$;

i.e., the side opposite an angle of 45° is equal to the hypotenuse divided by $\sqrt{2}$.

ERRATA.

Page	Line	For	Read.
38	12	changes	change
39	14	of.	cf.
65	8	XX	XX′
67	14	same	some
72	17	P	P_2
74	23	G	G'
124	13	2.6	2·6
137	17	from	form
145	10	circumference	circumferences
166	1	as it	as it is
178	6	$\div$	$+$